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THEORY OF ROLLING: SOLUTION OF THE COULOMB PROBLEM

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Abstract: A theory of rolling of round bodies in the normal mode with adhesion conditions satisfied on the entire contact area is proposed. This theory refines the classical Coulomb's theory of rolling in which the rolling moment is directly proportional to the pressing force (e.g., the weight of the rolling body). The rolling moment of cylinders is found to be directly proportional to the pressing force raised to a power of 3/2, and the rolling moment of balls and tori is proportional to the pressing force raised to a power of 4/3. It is shown that the normal mode of uniform rolling can only be provided for a certain ratio of the elastic constants of the materials of the round body and the base forming an ideal pair. The Coulomb problem is solved for the cases of rolling of an elastic cylinder over an elastic half-space, of an elastic ball over an elastic half-space, of an elastic torus over an elastic half-space, and of a cylinder and ball over a tightly stretched membrane. The rolling law is derived for such cases. The rolling friction coefficients, the rolling moment, and the rolling friction force are calculated.

Keywords: theory of rolling, Coulomb problem, rolling law, rolling moment, rolling friction coefficient, normal mode of rolling.

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1. FORMULATION OF THE PROBLEM

According to the classical Coulomb's law of rolling, the rolling moment of round bodies with a constant translational velocity is

$$M = TR = kN. \tag{1.1}$$

Here R is the radius of the round body (cylinder, wheel, ball, or torus), T is the thrust force applied to the round body center and equal to the resultant friction force on the contact area, N is the normal pressing force (e.g., the body weight) applied to the round body center, and k is the rolling friction coefficient, which is assumed to be a certain constant of the round body and base materials. Experimental determination of this coefficient is one of the basic problems of tribology.

The Coulomb law was refined in [1], where it was also demonstrated that the rolling friction coefficient of elastic round bodies in the normal mode of rolling is not a material constant; instead, it is a function of several parameters:

—for wheels and cylinders,

$$k = \eta_W (NRP)^{1/2}; \tag{1.2}$$

—for balls,

$$k = \eta_B (NRP)^{1/3}. \tag{1.3}$$

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Here η_W and η_B are the dimensionless rolling coefficients of cylinders (wheels) and balls, and P is the elastic compliance of the round body–base system. In the case of cylinders and wheels, N is the pressing force per unit length of the cylinder (wheel width).

As was shown in [1], in the normal mode of rolling of elastic bodies over an elastic base with adhesion conditions on the contact area, the value of k is equal to one half of the characteristic size of the contact area. Thus, solving the corresponding contact problem, we can use Eqs. (1.2) and (1.3) to calculate the dimensionless coefficients η_B and η_W, thus, finding a solution of the known Coulomb problem.

The Coulomb problem is solved below for the cases of rolling of an elastic cylinder over an elastic half-space, of an elastic ball over an elastic half-space, of an elastic torus over an elastic half-space, and of a cylinder and a ball over a tightly stretched membrane.

2. ROLLING OF AN ELASTIC CYLINDER

Let two elastic cylinders of radii R_1 and R_2 made of different materials contact each other under plane strain conditions, and let the width of the contact area $2a$ be much smaller than R_1 and R_2. Let also $R_2 > R_1$. We study the process of constant-velocity rolling of the cylinder of radius R_1 over the cylinder of radius R_2. The boundary conditions of the corresponding plane contact problem of the elasticity theory are formulated as follows:

$$y = 0, \quad |x| > a: \qquad (\sigma_y - i\tau_{xy})^{\pm} = 0; \tag{2.1}$$

$$y = 0, \quad |x| < a: \qquad [\sigma_y - i\tau_{xy}] = 0, \quad \Big[\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\Big] = -ix\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big). \tag{2.2}$$

Here x and y are the axes of the Cartesian coordinate system in a plane normal to the parallel axes of the cylinders (y is the axis of symmetry of the problem, and the origin is chosen to be at the center of the contact area), (u, v) is the displacement vector, σ_x, σ_y, and τ_{xy} are the stress tensor components, $A^{\pm} = \lim\limits_{z\to x\pm i0} A$, $[A] = A^+ - A^-$, and $z = x + iy$.

According to Eqs. (2.2), adhesion conditions necessary for the normal mode of rolling to occur are satisfied on the contact area. Let us use the Kolosov–Muskhelishvili presentations [2]:

$$\begin{gathered}\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\Phi_j(z) \qquad (j = 1, 2),\\ \sigma_y - i\tau_{xy} = \Phi_j(z) + \overline{\Phi_j(z)} + z\,\overline{\Phi_j'(z)} + \overline{\Psi_j(z)}\,,\\ 2\mu_j\Big(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\Big) = \varkappa_j\Phi_j(z) - \overline{\Phi_j(z)} - z\,\overline{\Phi_j'(z)} - \overline{\Psi_j(z)}\,.\end{gathered} \tag{2.3}$$

Here the subscripts $j = 1$ and $j = 2$ correspond to the upper and lower half-planes, $\Phi(z)$ and $\Psi(z)$ are functions of the complex variable, which are analytical in the corresponding half-planes, μ_j and ν_j are the shear modulus and Poisson's ratio of the corresponding material, and $\varkappa_j = 3 - 4\nu_j$.

Using the double analytical continuation

$$\begin{aligned}\Phi_1(z) &= -\overline{\Phi_1}\,(z) - z\,\overline{\Phi_1'}\,(z) - \overline{\Psi_1}\,(z) \qquad (y = \mathrm{Im}\,z < 0),\\ \Phi_2(z) &= -\overline{\Phi_2}\,(z) - z\,\overline{\Phi_2'}\,(z) - \overline{\Psi_2}\,(z) \qquad (y = \mathrm{Im}\,z > 0),\end{aligned} \tag{2.4}$$

we can formulate the boundary conditions (2.1) and (2.2) for the boundary-value Riemann problem on a cut $(-a, +a)$

$$\Phi_2^+ + m\Phi_2^- = \frac{2ix\mu_1\mu_2}{\mu_1 + \mu_2\varkappa_1}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \qquad (y = 0, \quad |x| < a), \tag{2.5}$$

where

$$m = \frac{\mu_2 + \mu_1\varkappa_2}{\mu_1 + \mu_2\varkappa_1}, \tag{2.6}$$

and $[\Phi_2(z)] = [\Phi_1(z)] = 0$ at $y = 0$, $|x| > a$, and

$$\Phi_1(z) = -\Phi_2(z). \tag{2.7}$$

The results of studying the solution of the boundary-value problem (2.5) show that it does not satisfy the fundamental condition of nonpenetration (see [1, 2]) at $m \neq 1$, i.e., the contact problem of adhesion of materials over the entire contact area is ill-posed. This means that slip zones are formed near the contact area ends at $m \neq 1$ [1, 2]. As a result, the nonpenetration condition is satisfied and the problem is no longer ill-posed.

It should be noted that the known condition of stress finiteness has no physical meaning and is usually not satisfied in problems with adhesion. A necessary condition is nonpenetration [1], i.e., two materials cannot be simultaneously located at one point of space (e.g., stamp and base materials).

A pair of materials satisfying the condition $m = 1$ is called an ideal pair because the presence of slip zones on the contact area leads to anomalous wear of materials during rolling and to anomalous rolling modes: emergency and accidental modes [1]. Therefore, from the viewpoint of applications, a pair of materials satisfying the condition $m = 1$ is an optimal one. The normal mode of rolling can be obtained only at $m = 1$.

Using Young's moduli E_1 and E_2, we can write the condition $m = 1$ as

$$\frac{E_1}{E_2} = \frac{1+\nu_1}{1+\nu_2}\,\frac{1-2\nu_1}{1-2\nu_2}. \tag{2.8}$$

In view of engineering capabilities of controlling elastic constants of composite materials, condition (2.8) is not a constraint; vice versa, it shows which elastic constants, e.g., a wheel should have for an optimal mode of rolling to be obtained.

In what follows, we study the process of cylinder rolling in the case of an ideal cylinder–base pair for which the normal mode of rolling can be ensured. The emergency and accidental modes of rolling are always accompanied by the stick-slip mode resulting in jerking motion [1].

At $m = 1$, the solution of the boundary-value problem (2.5) has the form

$$\Phi_2(z) = \frac{i\mu_1\mu_2}{\mu_1 + \mu_2\varkappa_1}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\Big(z - \sqrt{z^2 - a^2}\,\Big); \tag{2.9}$$

$$z \to \infty: \qquad \sqrt{z^2 - a^2} \to z, \quad \Phi_2(z) = iN/(2\pi z),$$

where

$$a^2 = \frac{1}{\pi}\,NP\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1}, \tag{2.10}$$

$P = 1/\mu_2 + \varkappa_1/\mu_1$ is the compliance of the elastic system, and N is the absolute value of the resultant pressure force on the contact area.

According to Eqs. (2.3), (2.4), and (2.9), the stresses on the contact area are determined by the formulas

$$\sigma_y = -\frac{2\mu_1\mu_2}{\mu_1 + \varkappa_1\mu_2}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\sqrt{a^2 - x^2}\,, \quad \tau_{xy} = 0 \qquad (y = 0, \quad |x| < a). \tag{2.11}$$

The shear stresses on the contact area appear at the instant of application of the tangential thrust force T, and a singularity of the shear stress arises on the edge of the contact area.

Cylinder rolling begins at the instant of application of this force to the center of the cylinder of radius R_1, when the end of the resultant force vector $(T, -N)$ reaches the end of the contact area and the instantaneous axis of rotation reaches the boundary of this area [1]. By virtue of the nonpenetration condition, application of the thrust force does not change the position and width of the contact area [1]. Based on Eq. (2.10), we obtain the following rolling law:

$$M = TR_1 = \frac{1}{\sqrt{\pi}}\,N^{3/2}\sqrt{P}\,\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/2}, \tag{2.12}$$

i.e., the rolling friction coefficient in the Coulomb law is

$$k = \frac{1}{\sqrt{\pi}}\,\sqrt{NP}\,\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/2}. \tag{2.13}$$

Two important particular cases of the solution of Eq. (2.12) should be noted.

1. Rolling of the cylinder over a half-space. In this case, we have $R_1 = R$ and $R_2 \to \infty$, and the rolling law is written as

$$M = TR = \frac{1}{\sqrt{\pi}} N^{3/2} \sqrt{PR} \qquad \Big(\eta_W = \frac{1}{\sqrt{\pi}}, \quad k = \frac{1}{\sqrt{\pi}} \sqrt{NPR}\,\Big). \tag{2.14}$$

2. Rolling of the cylinder over the surface of a cylindrical cavity in an elastic space. In this case, the curvature radius of the cavity in Eqs. (2.11) and (2.12) is negative, i.e., $R_2 < 0$. For instance, if the radii are close in value, i.e., $|R_2| = R_1(1 + \Delta)$, where $\Delta \ll 1$, then the rolling law is simplified:

$$M = TR_1 = N^{3/2} \sqrt{PR_1/(\pi\Delta)} \qquad (k = \sqrt{NPR_1/(\pi\Delta)}\,). \tag{2.15}$$

Formulas (2.12)–(2.15) refine the Coulomb rolling law.

3. ROLLING OF AN ELASTIC TORUS OVER AN ELASTIC HALF-SPACE

Let us consider a continuous elastic torus whose surface is formed by rotation of a circumference of radius r around an axis lying in the plane of this circumference, but not intersecting the latter, so that the circumference center during rotation forms another circumference of radius $R > r$. Let this torus press on the boundary of the elastic half-space $z < 0$ so that the plane of the major circle of the torus is perpendicular to the half-space boundary $z = 0$. The pressing force N is applied to the torus axis and is directed normal to the half-space boundary (it is transferred to the torus via spokes or a disk connecting the axis with the torus). Let us consider the corresponding contact problem of the pressure of an elastic torus on an elastic half-space, assuming that the shear stresses on the contact area are equal to zero.

In the vicinity of a small contact area, the torus surface coincides with the surface of an ellipsoidal paraboloid

$$z = \frac{x^2}{2(R + r)} + \frac{y^2}{2r},$$

whose directions and principal curvature radii at the initial contact point coincide with the directions and principal curvature radii of the torus r and $R + r$. The problem of the contact of two different smooth elastic paraboloids was solved in 1882 by Hertz [3]. According to Hertz's solution, a small contact area in the considered problem is the interior of the ellipse on the plane $z = 0$, which is the plane tangential to the paraboloids at the initial contact point:

$$x^2/L^2 + y^2/a^2 = 1 \qquad (L > a). \tag{3.1}$$

Here $2L$ is the major axis of the ellipse located on the x axis and $2a$ is the minor axis of the ellipse located on the y axis. The x and y axes coincide with the principal curvature axes of the torus at the initial contact point. During torus rolling, the point $(0, 0, R + r)$, which is the torus center, moves in the plane $y = 0$ parallel to the x axis. The stresses on the contact point are [3]

$$\sigma_z = -\frac{3N}{2\pi aL} \sqrt{1 - \frac{x^2}{L^2} - \frac{y^2}{a^2}}\,, \qquad \tau_{xz} = \tau_{yz} = 0. \tag{3.2}$$

According to Hertz's solution, the lengths of the ellipse semi-axes L and a, and also the vertical displacement of the torus w under the action of the force N are determined by the equations

$$(1 - e^2)\,\frac{D(e)}{B(e)} = \frac{r}{R + r}, \qquad 1 - e^2 = \frac{a^2}{L^2}; \tag{3.3}$$

$$L = \xi_L (NP)^{1/3} (R + r)^{1/3}, \qquad \xi_L = \Big(\frac{3}{2\pi}\, D(e)\Big)^{1/3}; \tag{3.4}$$

$$w = \xi_w \Big(\frac{NP}{\sqrt{R + r}}\Big)^{2/3}, \qquad \xi_w = \Big(\frac{9}{32\pi^2 D(e)}\Big)^{1/3} K(e); \tag{3.5}$$

$$P = \frac{1 - \nu_1}{\mu_1} + \frac{1 - \nu_2}{\mu_2} = 2\,\frac{1 - \nu_1^2}{E_1} + 2\frac{1 - \nu_2^2}{E_2},$$

where e is the ellipse eccentricity, P is the elastic compliance of the system, the subscripts 1 and 2 refer to the torus and base materials, and $K(e)$ and $E(e)$ are the total elliptical integrals:

$$K(e) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}, \qquad E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \varphi}\, d\varphi,$$

$$e^2 B(e) = E(e) - K(e)(1 - e^2), \qquad e^2 D(e) = K(e) - E(e).$$

Let us consider the most important limiting case of solution (3.3)–(3.5):

$$R \gg r, \quad r \gg a, \quad R \gg L, \quad 1 - e \ll 1, \quad L \gg a. \tag{3.6}$$

In this case, the contact area is an oblate ellipse, and the stress state in the plane $x = 0$ coincides with the stress state in the contact problem of the pressure of an elastic cylinder of radius r on an elastic half-space (see Section 2). In this limiting case, Eq. (3.3) acquires the form

$$m^2 \frac{\ln(4/m) - 1 - (1/4)m^2 \ln(4/m)}{1 + m^2 \ln(4/m) - m^2} = \frac{r}{R + r} \qquad \Big(m = \frac{a}{L}\Big). \tag{3.7}$$

At sufficiently small values of r/R, the asymptotic solution of Eq. (3.7) can be written as

$$\frac{a}{L} = \sqrt{\frac{r}{R + r}} \Big(\ln \sqrt{16 \frac{R + r}{r}} - 1\Big)^{-1} \qquad \Big(m = \frac{a}{L}, \quad e = \sqrt{1 - \frac{a^2}{L^2}}\Big). \tag{3.8}$$

In this case, Eqs. (3.4) and (3.5) take the form

$$L^3 = \frac{3}{2\pi} NP(R + r)\Big(\ln \frac{4}{m} - 1 - \frac{1}{4} m^2 \ln \frac{4}{m}\Big) \qquad \Big(m = \frac{a}{L}\Big); \tag{3.9}$$

$$w = \Big(\frac{9}{32\pi^2}\Big)^{1/3} \Big(\frac{NP}{\sqrt{R + r}}\Big)^{2/3} \frac{(1 - m^2)^{1/3}[(1 + m^2/4)\ln(4/m) - m^2/4]}{[(1 - m^2/4)\ln(4/m) - 1]^{1/3}}. \tag{3.10}$$

Let us consider the corresponding contact problem in the case where adhesion conditions necessary for the normal mode of torus rolling are satisfied on the entire contact area. Using the microscope principle, we can demonstrate that the nonpenetration condition is violated for an arbitrary pair of materials near the edge of the contact area if adhesion conditions are imposed on the entire contact area [2]. In other words, the normal mode of rolling of an elastic torus over an elastic base is impossible for an arbitrary pair of the torus and base materials: slip zones are formed near the edge of the contact area, leading to the emergency and accidental modes of rolling accompanied by the stick-slip mode and increased wear of materials [1].

It follows from Hertz's solution that adhesion conditions on the entire contact area and the normal mode of rolling can be obtained only for an ideal pair of the base and torus materials for which condition (2.8) is satisfied. For an ideal pair, the solution of the contact problem is Eqs. (3.2)–(3.10); moreover, the following adhesion condition is additionally satisfied on the contact area:

$$[u] = [v] = 0.$$

Here u and v are the components of the displacement vector along the x and y axes, respectively; the square brackets mean a jump in the quantity across the contact area. In what follows, we consider only an ideal pair of the torus and base materials for which the normal mode of rolling of an elastic torus over an elastic base is obtained.

In addition to the force N in the x direction, we apply a tangential thrust force T directed toward the torus center $(0, 0, R + r)$. The presence of this force induces shear stresses on the contact area, which are singular on the edge of this area, but does not affect the position and size of the contact area, because any change in the contact area with an unchanged value of N leads to violation of the fundamental condition of nonpenetration. The tangential thrust force does not affect the equilibrium state of the torus either, as long as the end of the vector (T, N) is located within the contact area. Under conditions of uniform rolling of the torus, the end of this vector is directed toward the point $(L, 0, 0)$ on the edge of the contact area.

In view of Eqs. (3.4), the law of torus rolling with a constant velocity is formulated as follows:

$$M = T(R + r) = ((3/(2\pi))PD(e))^{1/3} N^{4/3} (R + r)^{1/3},$$

i.e., the rolling friction coefficient in the Coulomb law is

$$k = ((3/(2\pi))NPD(e)(R+r))^{1/3}.$$

Here e is determined from Eq. (3.3).

In the case with $R \gg r$, by using Eq. (3.9), the law of torus rolling can be written as

$$M = T(R+r) = N^{4/3}\Big(\frac{3}{2\pi}\,P(R+r)\Big(\ln\frac{4}{m} - 1 - \frac{1}{4}\,m^2\ln\frac{4}{m}\Big)\Big)^{1/3},$$

where m is determined from Eq. (3.8).

4. ROLLING OF AN ELASTIC BALL OVER AN ELASTIC HALF-SPACE

Let us consider the case of rolling of an elastic ball of radius R_1 over another elastic ball of radius R_2, with $R_2 > R_1$. We use Hertz's solution for a particular case of a symmetric contact of two balls [3]:

$$a = \Big(\frac{3}{4}\,PN\Big)^{1/3}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/3} \qquad \Big(P = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\Big); \tag{4.1}$$

$$w = \Big(\frac{9}{16}\,N^2P^2\Big)^{1/3}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/3} \qquad (a \ll R_1). \tag{4.2}$$

Here a is the radius of the circular contact area, N is the resultant pressure force on the contact area directed along the axis of symmetry of the problem, and w is the distance between the ball centers after the contact beginning.

Hertz's solution implies that the shear stresses on the contact area are equal to zero. In the case of degeneration of ellipsoids into cylinders due to satisfaction of adhesion conditions on the contact area, the shear stresses there vanish at $m = 1$ (see [1] and Section 2). Under conditions of adhesion on the contact area, the problem of mutual local penetration of materials also arises in the problem of the contact of ellipsoids if $m \neq 1$. Therefore, this problem is ill-posed at $m \neq 1$.

We can demonstrate that the notion of an ideal pair of materials for which condition (2.8) is satisfied retains its physical meaning in the problem of the contact of elastic ellipsoids made of different materials: for an ideal pair, both the adhesion conditions and the condition of zero shear stresses are satisfied on the entire contact area. Therefore, for an ideal pair of materials, Eqs. (4.1) and (4.2) can also be used under conditions of adhesion on the contact area. Further in this section, we consider only the normal mode of ball rolling for an ideal pair of materials with adhesion conditions satisfied on the entire contact area.

Application of the tangential thrust force T to the center of the small ball induces shear stresses on the contact area, but does not affect the position and size of this area because the nonpenetration condition is satisfied. When the instantaneous axis of rotation of the small ball located in the plane of symmetry of the problem goes outside the contact area, the small ball starts rolling over the large ball. Thus, using Eq. (4.1), we obtain the following law of uniform rolling of the ball:

$$M = TR_1 = N^{4/3}\Big(\frac{3}{4}\,P\Big)^{1/3}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/3},$$

i.e., the rolling friction coefficient in the Coulomb law is

$$k = \Big(\frac{3}{4}\,NP\Big)^{1/3}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^{-1/3}.$$

In a particular case of rolling of an elastic ball over an elastic half-space with $R_1 = R$ and $R_2 \to \infty$, the rolling law for an ideal pair of materials takes the form

$$M = TR = N^{4/3}(3PR/4)^{1/3},$$

and

$$k = (3NPR/4)^{1/3}, \qquad \eta_B = (3/4)^{1/3}.$$

Obviously, it is necessary to know the rolling law to calculate the accelerated motion of round bodies under the action of the thrust force greater than the resultant shear friction force on the contact area T and also the decelerated motion of round bodies under the action of the inertial force and the thrust force, which is smaller than the resultant shear friction force on the contact area in this case. It is necessary to know the normal displacement w of the round body under the action of the force N to calculate both accelerated and decelerated motions of the body in the direction of the action of the force N (and rebound in the opposite direction).

5. ROLLING OF A CYLINDER AND A BALL OVER A MEMBRANE

Let us consider the case of cylinder and ball rolling over the surface of a flat momentless shell tightly stretched in all directions by the force $\Gamma = \sigma h$ (σ is the tensile stress and h is the shell thickness) acting in the plane of the shell.

5.1. Rolling of a Cylinder

Let us consider the contact of a cylinder $x^2+y^2 = R^2$ of radius R with a momentless film (membrane) under the action of the force N per unit length of the cylinder. Let $2a$ be the width of a small contact area $a \ll R$ and $a = R\sin\alpha$. The force N is applied to the cylinder center in the opposite direction to the y axis. Under conditions of adhesion between the cylinder and the film on the contact area, the force N is balanced by the film tension forces applied to the edge of this area; under slip conditions, it is balanced by the film pressure on the cylinder on the contact area.

The equation following from the energy conservation law is valid on the edge of the contact area [4, 5]:

$$\Gamma = \Gamma_c/(1-\cos\beta) \qquad (\beta < \alpha). \tag{5.1}$$

Here Γ_c is the specific energy of adhesion of the film and cylinder materials and β is the angle between the film and the plane tangential to the cylinder on the edge of the contact area. For a small contact area, with small values of α and β, Eq. (5.1) has the form

$$\beta^2 = 2\Gamma_c/\Gamma. \tag{5.2}$$

For a tightly stretched film, we have $\Gamma \gg \Gamma_c$, more exactly, $\Gamma\alpha^2 > 2\Gamma_c$.

Thus, the film forms an angle $\alpha-\beta$ with the x axis (rolling direction); therefore, under conditions of adhesion between the film and the cylinder, the equilibrium equation is written as

$$N = 2\Gamma\sin(\alpha-\beta); \tag{5.3}$$

at small values of α and β, this equation is

$$N = 2\Gamma(\alpha-\beta). \tag{5.4}$$

At $\Gamma_c = 0$ (when $\beta = 0$), the equilibrium equations under slip and no-slip (adhesion) conditions coincide.

Let the thrust force T (per unit length of the cylinder) be applied to the cylinder center in the direction of the x axis. The cylinder starts rolling when the end of the vector $(T, -N)$ of the resultant force reaches the edge of the contact area. Thus, using Eqs. (5.2) and (5.4), we obtain the following law of uniform rolling of the cylinder over the membrane:

$$M = TR = NR(N/(2\Gamma) + \sqrt{2\Gamma_c/\Gamma}\ ), \tag{5.5}$$

i.e., the rolling friction coefficient in the Coulomb law is

$$k = R(N/(2\Gamma) + \sqrt{2\Gamma_c/\Gamma}\ ). \tag{5.6}$$

5.2. Rolling of a Ball over a Membrane

Let a ball $x^2+y^2+z^2 = R^2$ press a membrane with a force N applied at the ball center in the opposite direction to the z axis, which is the axis of symmetry of the problem. A small contact area forms a circle of radius a, and $a = R\sin\alpha$. As in the case of cylinder rolling, the film forms an angle $\alpha-\beta$ with the xy plane on the edge of the contact area with the ball [the angle β is determined from Eqs. (5.1) and (5.2)]. Under conditions of adhesion of the film and ball on the contact area, the force N is balanced by the film tension forces uniformly distributed over the edge. In this case, the equilibrium equation for the ball on the contact area is written as

$$N = 2\pi a\Gamma(\alpha-\beta) \qquad (\alpha > \beta). \tag{5.7}$$

Thus, according to Eqs. (5.2) and (5.7), the law of uniform rolling of a ball over a tightly stretched membrane has the form

$$M = TR = NR\Big(\sqrt{\frac{N}{2\pi R\Gamma} + \frac{\Gamma_c}{2\Gamma}} + \sqrt{\frac{\Gamma_c}{2\Gamma}}\,\Big), \tag{5.8}$$

and the friction rolling coefficient in the Coulomb law is

$$k = R\Big(\sqrt{\frac{N}{2\pi R\Gamma} + \frac{\Gamma_c}{2\Gamma}} + \sqrt{\frac{\Gamma_c}{2\Gamma}}\,\Big) \qquad (\Gamma \gg \Gamma_c). \tag{5.9}$$

The rolling laws (5.5), (5.6) and (5.8), (5.9) are the solution of the Coulomb problem in the case considered here. These laws are necessary to study the dynamics of motion of round bodies.

CONCLUSIONS

The Coulomb problem was first formulated at the end of the 18th century, when Charles Coulomb derived the rolling law with the use of an empirical coefficient of rolling friction. In this paper, this coefficient calculated for the cases of rolling of elastic cylinders, balls, and tori over an elastic half-space is a function of the basic parameters of the contacting bodies. The law of cylinder and ball rolling over a tightly stretched membrane is also derived.

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