Серия: Инвариантные интегралы

An Invariant Integral Series

Тематика публикации: Применение инвариантных интегралов в физике и механике

POINT DEFECTS IN SOLIDS

Genady P. Cherepanov

ТОЧЕЧНЫЕ ДЕФЕКТЫ В ТВЁРДЫХ ТЕЛАХ

Г. П. Черепанов

Точечными дефектами называются инородные включения и пустоты (дырки), размер которых пренебрежимо мал по сравнению с характерными размерами тела. Такие дефекты присутствуют во всех конструкционных материалах, также как дефекты типа дислокаций и мелких трещин в нано или микро- масштабах. Развитие и взаимодействие всех этих дефектов материала приводит к таким свойствам его поведения в макромасштабе как пластичность, ползучесть, упрочнение, старение и всем другим свойствам неупругости. Поэтому теория твердых тел, по существу, сводится к изучению проблем развития и взаимодействия множества указанных дефектов в упругом теле.

Каждый дефект создаёт некоторое поле напряжений и деформаций в упругом теле, характеризующееся определённой асимптотикой на расстояниях, больших по сравнению с размером дефекта, но малых по сравнению с характерными размерами образца или конструкции. Если размер дефекта очень мал, то такой дефект можно описать этой асимптотикой поля, имеющей особенность (сингулярность) в точке расположения дефекта, как это принято в физике элементарных частиц. Точечный дефект со своим собственным полем становится своеобразной элементарной частицей. Взаимодействие, слияние/деление и перемещение дефектов, включая дислокации и трещины, в макромасштабе приводит к пластичности, ползучести и всем другим типам неупругого поведения упругого материала.

Автор начал заниматься проблемой точечных дефектов в 1970-ых годах по предложению Джока Эшелби (Jock Eshelby), который в это время начал заниматься «моими» трещинами, а до того занимался дислокациями и включениями в упругом теле. Используя инвариантные интегралы, мне удалось найти силы взаимодействия точечных дефектов типа включений и дырок, а затем рассчитать также влияние этих точечных дефектов и любых континуальных распределений множества таких дефектов на силы, движущие дислокации и трещины. В макромасштабе найденные формулы описывают увеличение или уменьшение предела текучести (hardening or softening) упрочняющегося упруго-пластического материала, а также увеличение или уменьшение вязкости разрушения материала (strengthening or embrittlement). Удалось найти также пути (траектории) виртуальной миграции точечных дефектов, которая в макромасштабе описывает ползучесть и повреждённость (несплошность)материала. Результаты этих исследований кратко изложены в приводимой ниже главе 5 «Point defects in solids», взятой из книги «METHODS OF FRACTURE MECHANICS: Solid Matter Physics». По предложению Бруса Билби (Bruce Bilby) эти результаты были опубликованы ранее в мемориальном томе, посвящённом памяти Эшелби (см. ссылку [7] в этой главе).

В связи с настоящей публикацией следует остановиться на том, что модно называть повреждённостью. По Л.М.Качанову, повреждённость или несплошность материала есть неопределённый параметр, изменяющийся от нуля до единицы, так что нуль соответствует неповреждённому материалу, а единица---разрушенному. Аналогичное определение дано Ю.Н.Работновым. Вспоминаю разговор с Качановым в 1963 году, которого я случайно встретил в Киеве.

Я спросил: «Лазарь Маркович, в науке принято иметь дело с измеримыми величинами, которые можно определить/измерить хотя бы в мысленном эксперименте. Вашу повреждённость, как святой дух, невозможно измерить, поскольку она не определена. А уравнений, математически описывающих изменение этого параметра во времени, может быть столько же, сколько исследователей.» Качанов ответил: «В будущем, надеюсь, найдут, как её измерить.» Аналогичный обмен мнениями произошёл позднее с Работновым.

Хотя проблема измерения при таком подходе осталась неразрешимой, соблазн создания «своей» теории повреждённости простыми математическими манипуляциями оказался так велик, что у подхода Качанова-Работнова оказалось множество последователей. Удивляться этому не следует: учёные---тоже люди, и как большинство людей, подвержены религиозным верованиям, закладываемым изначале авторитетными индивидуумами. В США, например, имеется свыше 200 различных религиозных верований, официально зарегистрированных. Кстати, согласно исламу и иудаизму, как известно, бог (аллах) не имеет каких-либо физических измерений, а изображение бога в виде какого-то человека считается святотатством (так как тем самым поклоняются этому реальному человеку). Христос и отец-бог христиан изображаются по-разному, в зависимости от художника, и только святой дух не имеет физических измерений. В этом смысле ислам и иудаизм---более совершенные религии по сравнению с христианством.

Не следует думать, что среди учёных атеисты встречаются чаще, чем среди людей, далёких от науки. Наоборот, большинство великих учёных были глубоко верующими людьми---и это не парадокс: твёрдая вера---та черта характера, которая совершенно необходима настоящему учёному. Открытие вначале всегда---некое необъяснимое чудо, мимо которого человек с нетвёрдой верой пройдёт и не поверит---и такое нередко случалось в истории науки. Лейбниц и Ньютон были не просто верующими людьми, но и выдающимися богословами своего времени, а собственных богословских сочинений у Ньютона было больше, чем научных.

Поэтому увлечение большой группы учёных подходом Качанова-Работнова к проблеме повреждённости материала следует считать безвредной разновидностью шаманства или верования, неким парапсихологическим увлечением. Кстати, рассмотрение ниспадающей ветви на диаграмме «напряжение-деформация» упруго-пластических тел, с негативными модулями и тому подобной «информацией», также относится к увлечениям парапсихологией, привлекающим не один десяток учёных, в том числе, самых великих учёных нашего времени.(Подробнее, о последнем см. мою заметку в журнале «Прикладная механика»,том 46, № 2, стр.138-142, 2010.) Научное описание повреждённости основано на изучении поведения множества дырок (пустот), описанных в публикуемой ниже главе, а мелкие трещины, в особенности крестообразные, сводятся асимптотически к некоторым эквивалентным точечным дыркам.

При подготовке этой электронной публикации автором исправлены замеченные опечатки издания 1997 года.

SOLID MECHANICS AND ITS APPLICATIONS

G.P. Cherepanov

Methods of Fracture Mechanics: Solid Matter Physics

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Methods of Fracture Mechanics: Solid Matter Physics

G.P. Cherepanov

Modern fracture mechanics considers phenomena at many levels, macro and micro; it is therefore inextricably linked to methods of theoretical and mathematical physics. This book introduces these sophisticated methods in a straightforward manner. The methods are applied to several important phenomena of solid state physics which impinge on fracture mechanics: adhesion, defect nucleation and growth, dislocation emission, sintering, the electron beam effect and fractal cracks. The book shows how the mathematical models for such processes may be set up, and how the equations so formulated may be solved and interpreted. The many open problems which are encountered will provide topics for MS and PhD theses in fracture mechanics, and in theoretical and experimental physics.

As a supplementary text, the book can be used in graduate level courses on fracture mechanics, solid matter physics, and mechanics of solids, or in a special course on the application of fracture mechanics methods in solid matter physics.

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Methods of Fracture Mechanics: Solid Matter Physics

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ОГЛАВЛЕНИЕ

ГЛАВА 7. РЕЛЯТИВИСТСКИЕ ЭЛЕКТРОННЫЕ ЛУЧИ В ТВЕРДЫХ МАТЕРИАЛАХ

ГЛАВА 8. ФРАКТАЛЫ В МЕХАНИКЕ РАЗРУШЕНИЯ

ПРЕДМЕТНЫЙ ИНДЕКС

ПРЕДИСЛОВИЕ АВТОРА

Метод—это способ получения результата. Не владея методом, можно надеяться только на запоминание результата, полученного кем-то. А использование результата, не зная как он получен и, следовательно, не зная границ справедливости этого результата, обычно ведет к крупным ошибкам. Поэтому методы составляют наиболее существенную часть любой дисциплины. Эта книга - для тех трудолюбивых студентов, которые желают быстро и глубоко овладеть методами механики и физики разрушения. Прежде чем читать главу, посмотри на задачи, помещенные в её конце и попробуй решить их. Если ты сможешь это сделать, можешь не изучать эту главу.

Степень овладения методами главы пропорциональна числу решённых задач. Это ---главный показатель успешного изучения главы. Самой большой похвалой книге была бы приобретенная способность читателя решить все эти задачи. Как правило, метод излагается на некоторых характерных примерах, обычно более простых, чем соответствующие проблемы в конце главы. Чтобы помочь решению последних, даются краткие, но иногда довольно подробные подсказки. Таким образом, задачи, приведённые в книге, составляют её неотъемлемую часть и играют решающую роль в освоении излагаемых методов. Такой подход позволяет быстро овладеть сложным методом, но требует интенсивной и напряжённой работы читателя. В качестве примера аналогичного подхода можно привести книги Л.Д.Ландау и Е.М. Лифшица по теоретической физике. Систематическое изложение решений всех приведенных в книге задач потребовало бы многих томов. Некоторые идеи книги могут служить темами кандидатских и докторских диссертаций в механике разрушения и физике твёрдого тела. Поэтому она будет полезна как для преподавателей, так и для студентов, желающих получить учёную степень в этой области.

Каждая дисциплина использует известные методы, приспособленные к ее специфическим требованиям, а также развивает свои собственные методы, которые могут проникать в другие области и влиять на другие дисциплины. В книге излагаются некоторые приложения методов механики разрушения в физике твёрдого тела. В дальнейшем планируется приложение этих методов к проблемам механики твёрдого тела, к теории материалов, их обработки и изготовления, к оптимальному проектированию конструкций и материалов, к оценке и обеспечению безопасности конструкций и строений, к проблемам технологии и геофизики.

Книга обращается к наиболее трудным и интригующим темам физики твёрдого тела, интересным как для теории, так и для приложений. В качестве дополнительного источника книга может использоваться преподавателями и аспирантами в основных курсах (механики разрушения, физики и механики твёрдого тела), или в качестве основного источника в специальном односеместровом курсе по теме главы. Для удобства чтения литературные ссылки помещены в конце главы. Каждая глава начинается с короткой преамбулы, указывающей главные методы и подходы в данной главе. Для того, кто заинтересован, в основном, в методах, рекомендуется просмотреть вначале преамбулу.

Основные методы, используемые и развитые в книге, следующие:

---- математические методы моделирования, включая метод идентификации, метод аналогий и принцип Маха («экономии мышления»), а также эмпирические методы;

---- методы решения обыкновенных дифференциальных уравнений и уравнений в частных производных, функциональных и конечно-разностных уравнений дискретной математики, сингулярных интегральных и интегро-дифференциальных уравнений, которые возникают в различных краевых задачах механики разрушения;

---- методы характеристик, интегральных преобразований, автомодельных решений, сращивания асимптотических разложений, метод Смирнова-Соболева решения волновых уравнений, метод пограничного слоя, метод возмущения, метод Винера-Хопфа и Нобла-Джонса, метод Гахова решения проблем Римана-Гильберта и др.;

---- методы, развитые впервые в механике разрушения: метод инвариантных интегралов, метод сингулярных и обобщенных решений, метод движущих и конфигурационных сил;

---- методы фрактальной геометрии, теории катастроф или сингулярных преобразований;

---- численные методы, включая различные модификации метода конечных элементов и метода конечных разностей.

Наиболее важной стадией в изучении любой проблемы является формулировка «хорошей» математической модели, предназначенной для решения проблемы. От исследователя эта стадия требует знания некоторого критического количества основной информации относительно изучаемого явления, логический всесторонний анализ этой информации и, в результате, интуитивную идентификацию подходящей математической модели. В зависимости от целей исследования одна модель может быть лучше другой. Изобретение и использование различных моделей зависит от

искусства и мастерства экспертов в любой области знания. «Хорошая» модель должна быть:
---- логически последовательной, то есть не противоречащей сама себе;
---- адекватной изучаемому явлению;
---- прогнозирующей, то есть способной к предсказанию, с необходимой точностью, событий и ситуаций вне первоначального банка данных;
---- последовательной, то есть не противоречащей апробированным моделям в общей области их действия;
---- поддающейся проверке, то есть способной к проверке испытанием или мысленным экспериментом;
---- практической, то есть простой и удобной для применения;
---- экономичной, то есть содержать необходимый минимум новых постоянных;
---- и, наконец, изящной, то есть доставляющей эстетическое удовлетворение её изобретателю и пользователям.

Как правило, независимые исследователи, изучающие одну и ту же проблему, изобретают различные модели, но только одна из них выживает. Модели могут быть более или менее хороши в зависимости от глубины и широты их предсказаний. Обычно возможны многие модели описываемого явления, но только одна из них — лучшая. Создание и поиск лучших моделей — цель любой науки или области знания. Тот, кто знает методы лучше, более успешен и в изобретении лучших моделей.

Механика разрушения оказала влияние на основы физики твёрдого тела. В настоящей книге рассматривается применение основных методов и идей механики разрушения к следующим проблемам физики твёрдого тела:
--- поверхностная энергия и энергия адгезии (сцепления) различных твёрдых материалов (глава 1);
---- флюктуации и кинетическая теория разрушения (глава 2);
---- механизмы зарождения трещин (глава 3);
---- спекание порошков и создание материалов (глава 4);
---- точечные дефекты в материалах, их взаимодействие и движение (глава 5);
---- эмиссия дислокаций в кристаллах и наномеханика разрушения (глава 6);
---- релятивистские электронные лучи в диэлектрических материалах, взаимодействие электронов и разрушение материалов (глава 7);
---- фракталы в механике разрушения (глава 8).

Более подробная информация о содержании глав дана в оглавлении.

В создание этой книги были, прямо или косвенно, вовлечены многие. Моя жена Лариса Черепанова напечатала большую часть книги и подготовила фигуры. Мои старшие коллеги и прежние друзья, прежде всего Л.А.Галин, В.В. Соколовский, Г.И.Баренблатт и Л.И.Седов, повлияли на мое мышление и стиль работы. Джок Эшелби, Джон Гилман, Джим Райс, Берни Будянский, Джон Хатчинсон и другие предложили конструктивную критику некоторых тем книги. Приятная обязанность — выразить всем им мою искреннюю благодарность. Особо следует поблагодарить редактора серии книг по механике, проф. Гладвелла, который был первым читателем этой книги. Элен Руни, Кэти Рустад, Мэгги Забало и Кэролайн Либерти оказали помощь при оформлении рукописи. Многие другие индивидуумы, вовлеченные в этот проект, указаны на соответствующих страницах книги или в библиографии.

Некоторые исследования, приведенные в книге, в течение 1991-95 гг. были финансированы следующими грантами: NSF Grant MSS 9224936, NASA Grant 541803700, AFOSR Grant 59107 и U. S. Army SSDC Grant DASG 60-94-C-0032.

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AUTHOR'S PREFACE

A method is a way of achieving a result. Without mastering a method, one can only hope to memorize the result. Using the result without proper knowledge of the underlying method and hence neglecting the limitations of this result usually leads to terrible mistakes. Therefore, methods form a most substantial part of a discipline. This book is for those hard-working students who desire to master the methods for the shortest time. Look at the problems placed at the end of a Chapter and try to solve them. If you can solve these problems without reading the Chapter, you do not need to scrutinize it.

Your skills in mastering the methods of the Chapter are proportional to the number of problems you can solve. This is the main indicator of a successful study of the Chapter. The highest praise of the book would be an acquired ability of the reader to solve all these problems. The methods are treated on some particular exemplar cases, which are usually simpler than the corresponding problems at the end of Chapters. To help to solve the latter, hints, sometimes in many details, are provided. So, the Chapter problems constitute an inalienable part and play a decisive role in the book. This approach allows one to get into the complicated methods in a straightforward manner, but it requires an intense work of the reader to solve each Chapter's problems. As an example of the similar approach or style, the well-known series of books on theoretical physics by Landau and Lifshitz can be mentioned. A systematic treatment of all these methods would necessitate dozens of volumes. The solutions manual is available to qualified users. The book is rich in topics for MS and PhD theses in the area of solid and fracture mechanics and physics. Therefore, it will be useful for a teacher and one who seeks a degree in this field.

Every discipline uses the known methods accommodated to its specific demands and develops its own methods that penetrate and influence other disciplines. The present volume is dedicated to some applications of the methods of fracture mechanics in solid matter physics. It is planned that the following volumes will be dedicated to some applications of these methods in mechanics of solids, theory of materials, structural integrity, material processing and manufacturing, optimal design, technology, and geophysics.

The topics are selected from the most challenging problems that are interesting for applications. As a supplementary text, the book can be used by teachers and graduate students in core courses on fracture mechanics, solid matter physics, and mechanics of solids, or in a special course on the topic of a Chapter. For convenience of reading, pertinent references are placed at the end of every Chapter. Every Chapter begins with a short preamble indicating the main methods and procedures treated in the Chapter. For those who are interested mostly in methods, it is recommended that, at first, they look through the preambles.

The basic methods used and developed in the book are as follows:

- **Mathematical modeling** methods including identification methods, the method of analogy, trial-and-error methods, and Mach's "thought economy" principle;
- Methods of solving **ordinary and partial differential equations, functional and finite difference equations of discrete mathematics, singular and regular integral equations, and integro-differential** equations which appear in various boundary value problems of fracture mechanics ;

- Methods of solving miscellaneous boundary value problems of fracture mechanics including:asymptotic **methods of boundary layer, the method of self-similar - solutions, the matched asymptotic expansions, the method of characteristics, integral transform methods, the Wiener-Hopf and Noble-Jones methods of reduction to the Riemann and Hilbert problems** of the theory of functions of a complex variable, and the **perturbation method**;
- Method of **invariant integrals,** method of **singular and generalized solutions,** and method of **driving or configurational forces**, which were first developed in fracture mechanics;
- Methods of **fractal geometry, catastrophe theory or singular mappings;**
- **Numerical methods** including finite element and finite difference methods.

The most important stage in the study of a problem is the formulation of a predictive model; it should properly address the main challenges of the problem. From an explorer, this stage requires the knowledge of a certain critical amount of specific basic information about the underlying phenomenon (original data base), a logical comprehensive analysis of the information (data analysis), and as a result, an intuitive finding of the mathematical model which can properly describe the main features of the phenomenon. Depending on the goals of an investigation one model can fit better than another one. Inventing and using different models make up the arts and skills of experts in any field of knowledge. In brief, a good model should be:

- Logically self-consistent, that is, not contradicting itself;
- Adequate to a corresponding real phenomenon;
- Predictive, that is, capable of predicting events and situations outside an original data basis, within the limits of necessary accuracy;
- Consistent, that is, not contradicting the approved models in a common area of their action;
- Verifiable, that is, capable of being checked by test or mind;
- Practical, that is, simple and convenient for use;
- Economical, that is, containing a necessary minimum of new constants; and last but not the least
- Beautiful, that is, delivering aesthetic emotions to the inventor and users.

As a rule, independent explorers studying the same problem invent different models, and only one of them survives. A model can be powerful or less powerful depending on the breadth and depth of its predictions. It is recognized that many models can possibly describe a phenomenon, but there is only one which is best. Creation and search for the best models is the final goal of any science or knowledge. Those who know methods better are more successful in the model invention.

Fracture mechanics has influenced the fundamentals of solid matter physics. In this book we consider the application of basic methods and ideas of fracture mechanics to the following problems of solid matter physics.

- Surface energy of solids (Chapter 1);
- Fluctuations and the kinetic theory of fracture (Chapter 2);
- Crack nucleation (Chapter 3);
- Physics of sintering (Chapter 4);

- Point defects in solids (Chapter 5);
- Dislocation emission (Chapter 6);
- Relativistic electron beams in a solid (Chapter 7);
- Fractals in fracture of solids (Chapter 8).

More detailed information about the Chapter is given in Table of Contents.

Many people have, directly or indirectly, been involved in the creation of the book.

My wife Larisa Cherepanov typed most part of the book and prepared figures. Without her devotion to the project, the book could not have been written. My senior colleagues and former friends, first of all Lev Galin, Vadim Sokolovsky, Grigory Barenblatt and Leonid Sedov, influenced my way of thinking and style of work. Jock Eshelby, John Gilman, Jim Rice, Bernie Budiansky, John Hutchinson and others offered constructive criticism of some topics of the book. It is my duty to express my sincere gratitute to all of them .

My special thanks go to Professor Gladwell, Editor of this series, who generously read a draft of the entire manuscript. The help of the staff of the Mechanical Engineering Department, namely, Helen Rooney, Cathy Rustad, Maggie Zabalo and Caroline Liberty, is gratefully acknowledged. Many other individuals involved in the project are identified on the relevant pages of the book or in the bibliography.

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CHAPTER 5

POINT DEFECTS IN SOLIDS

The properties of solid matter depend significantly on the distribution, mobility and mode of point defects like interstitial atoms, vacancies and holes. The methods of fracture mechanics enable us to derive the forces driving the point defects, study their motion, and thus predict the corresponding response of the solid matter. The technique of invariant integrals is extensively explored in this chapter for the derivation of the defect-driving forces, and the solution of some problems relating to the motion of the defects. The method of invariant integrals introduced earlier by the present author is treated here as an alternative approach to the basic classic theories of mathematical physics including gravitational theory, electromagnetic field theory, nonequilibrium thermodynamics, gas dynamics and elasticity theory. Arts of the formulation of well-posed governing equations of the physical problems and skills for their solution are also developed in this chapter.

5.1. Conservation Laws and Invariant Integrals

Energy, mass, charge, momentum, entropy, moment of momentum and other properties of a continuous medium may be subject to conservation laws. The law of conservation can be written in the form of the sum of the non-steady and convective components as

$$\int_V \frac{\partial F_{\mathrm{v}}}{\partial t}\, dV + \int_\Sigma G_i\, n_i d\Sigma = 0 \qquad (i = 1,2,3)\,. \tag{5.1}$$

Here, $F_{\mathrm{v}}\ (x_1, x_{2,}, x_3\ , t)$ and $G_i\ (x_1, x_2, x_3\ , t)$ are some functions depending on the field state variables specified in the reference frame $0(x_1x_2x_3)$ of Cartesian coordinates x_1, $\mathrm{x}_{2,}$, x_3 and time t, V is the volume domain in $0(x_1x_2x_3)$ bounded by the surface Σ, and n_1, n_2 and n_3 are the outer unit normal vector components on Σ. The coordinate frame $0(x_1x_2x_3)$ can be fixed or moving with a constant velocity with respect to a fixed coordinate frame. The repeated index denotes summation over the index (for example, $G_in_i = G_1\, n_1 + G_2\, n_2 + G_3\, n_3$). The quantities F and/or G_i can be some components of a vector and/or a tensor.

According to Gauss - Ostrogradsky's divergence theorem,

$$\int_\Sigma A\, n_i\ d\ \Sigma = \int_V A_{,i}\ dV \quad , \tag{5.2a}$$

$$\int_\Sigma A_i\ n_i\ d\ \Sigma = \int_V A_{i\ ,i}\ dV \quad , \tag{5.2b}$$

$$\int_{\Sigma} A_{ik}\ n_i\ d\ \Sigma = \int_{V} A_{ik,i}\ dV \qquad (i = 1,2,3), \qquad (5.2c)$$

and so on.

Here, A, A_i, and A_{ik} are the scalar, vector and second-rank tensor functions correspondingly. The comma with a succeeding index denotes the partial derivative with respect to the corresponding coordinate (for example, $A_{,i} = \partial A/\partial x_i$).

We apply the divergence theorem to the second term in Equation (5.1) and reduce this equation to

$$\int_{V} \left(\frac{\partial F_{\mathrm{v}}}{\partial t} + G_{i,i} \right) dV = 0 \qquad (i = 1,2,3) \quad , \qquad (5.3a)$$

and then derive the conservation law in the local form,

$$\frac{\partial F_{\mathrm{v}}}{\partial t} + G_{i,i} = 0 \qquad (i = 1,2,3) \qquad (5.3b)$$

because V is arbitrary. This is the common way governing equations are obtained.

This local formulation of the conservation law has some disadvantages, because the derivatives lose sense at the singular points, lines and surfaces of the physical field. Meanwhile, the study of the singularities is the principal interest for a theory and its applications. The singularities of the field and the corresponding singular solutions contain the basic information about the advantages and disadvantages of the theory, the limits of its applicability, and its opportunities in practice. Therefore, it is important to have other formulations of governing equations of mathematical physics which would hold at both regular and singular points of the field. Such a formulation can be provided by invariant integrals.

Introduce the following potential, $\Pi(x_1, x_2, x_3, t)$, by

$$\frac{\partial F_{\mathrm{v}}}{\partial t} = \Pi_{i,i} \qquad (5.4)$$

Substituting $\partial F_v / \partial t$ in Equation (5.1) by Equation (5.4) and applying the divergence theorem we find,

$$\int_{\Sigma} \left(G_i + \Pi_i \right) n_i\ d\Sigma = 0 \qquad (5.5)$$

The integral in (5.5) is invariant with respect to Σ; therefore, it is called the invariant integral* .

Equation (5.5) is obviously equivalent to Equation (5.3b) at regular points, and to Equation (5.1) at any point. These are different mathematical formulations of one and the same law of conservation.

Consider a point singularity of the field at point 0 that can move in the reference frame $0(x_1x_2x_3)$ fixed at 0. Suppose that the singularity generates or absorbs the property which is conserved at the regular points according to the conservation law. Hence, the singular point can be considered as a source or sink of magnitude Γ,

$$\Gamma = \lim_{\Sigma \to O} \int_{\Sigma} \left(G_i + \Pi_i \right) n_i \, d\Sigma \quad , \tag{5.6}$$

according to Equation (5.5). In (5.6), closed surface Σ shrinks to 0.

Equations (5.5) and (5.6) are especially simple when the field is steady-state in the moving coordinate frame $0(x_1\ x_2\ x_3\ )$ or stationary in the fixed coordinate frame $0(x_1x_2x_3)$. In this case, $\Pi_i = 0$, and the G_i do not depend on t, so that Equations (5.5) and (5.6) are reduced to:

$$\int_{\Sigma} G_i n_i d\Sigma = 0 \quad \text{and} \quad \Gamma = \lim_{\Sigma \to 0} \int_{\Sigma} G_i n_i d\Sigma \tag{5.7}$$

Let us consider the basic fields of mathematical physics.

Gravitational field

Let the field of gravitation be defined by a potential $\varphi(x_1, x_2, x_3)$ satisfying the energy conservation law written in the form of the following invariant integrals

$$\frac{1}{4\pi f} \int_{\Sigma} \left(\frac{1}{2} \varphi,_i \varphi,_i n_k - \varphi,_i \varphi,_k n_i \right) d\Sigma = 0 \tag{5.8a}$$

$$\frac{1}{4\pi f} \lim_{\Sigma \to O} \int_{\Sigma} \left(\frac{1}{2} \varphi,_i \varphi,_i n_k - \varphi,_i \varphi,_k n_i \right) d\Sigma = \Gamma_k \quad , \tag{5.8b}$$

where f is a constant, and Γ_k is the energy of the field spent to move the point singularity 0 unit length distance along the x_k axis, that is, Γ_k , is the kth component of the driving force.

* This name was introduced by the present author who advanced the general method. The path-independent integral is the name introduced by Eshelby for some integrals of this kind in elasticity theory.

We show that this formulation of the gravitation theory is equivalent to the Newtonian theory. By means of the divergence theorem, we transform Equation (5.8a) to

$$\begin{aligned}&\int_{\Sigma} \left(\frac{1}{2}\varphi,_i \varphi,_i n_k - \varphi,_i \varphi_k n_i\right) d\Sigma = \\ &= \int_V \left[\frac{1}{2}(\varphi,_i \varphi,_i),_k - (\varphi,_i \varphi,_k),_i\right] dV = \\ &= \int_V (\varphi,_{ik} \varphi,_i - \varphi,_i \varphi,_{ki} - \varphi,_{ii} \varphi,_k)\, dV = -\int_v \varphi,_{ii}\varphi,_k \, dV = 0\end{aligned} \tag{5.9}$$

Because V is arbitrary, φ satisfies Laplace's equation, $\varphi,_{ii} = 0$, that is, φ is a harmonic function.

Consider a point source of the field at the coordinate origin, 0. The simplest solution to Laplace's equation, singular at 0, is:

$$\varphi = -f\,\frac{M}{r} - g_k\, x_k \qquad (r^2 = x_i x_i)\quad , \tag{5.10}$$

where M and g_k are some constants characterizing the strength of the source and intensity of the external field at 0, respectively. By repeated differentiation with respect to x_j one can obtain all the solutions to Laplace's equation which are singular at 0.

Let 0 be inside Σ. Using the rule of Γ- integration[6] we calculate Γ_k in Equation (5.8b) over the faces Σ of the narrow parallelepiped along $x_1 = \pm L$, $x_2 = \pm L$, $x_3 = \pm\,\delta$:

$$\begin{aligned}\Gamma_1 &= \frac{1}{2\pi f} \lim_{\substack{\delta/L\to 0\\ L\to 0}} \int_{-L}^{+L}\int_{-L}^{+L} (\varphi,_1 \varphi,_3)\Big|_{x_3=\delta} dx_1\, dx_2 = \\ &= \frac{1}{2\pi f} \lim_{\substack{L\to 0\\ \delta/L\to 0}} \int_{-L}^{+L}\int_{-L}^{+L} \left(g_1 \frac{fM}{r^3}\delta + g_3\frac{fM}{r^3}x_1\right) dx_1 dx_2 = \\ &= \frac{g_1 M}{2\pi} \lim_{\substack{L\to 0\\ \delta/L\to 0}} \int_{-L}^{+L}\int_{-L}^{+L} \frac{\delta\, dx_1 dx_2}{\left(x_1^2 + x_2^2 + \delta^2\right)^{3/2}} = \frac{g_1 M}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dt_1 dt_2}{\left(1+t_1^2+t_2^2\right)^{3/2}} \quad .\end{aligned} \tag{5.11}$$

Using polar coordinates R and θ, where $R^2 = t_1^2 + t_2^2$ and $dt_1\, dt_2 = Rd\theta dR$, we find

$$\Gamma_1 = \frac{g_1 M}{2\pi}\int_0^{2\pi} dx \int_0^{\infty} \frac{RdR}{\left(1+R^2\right)^{3/2}} = g_1 M \quad . \tag{5.12}$$

The analogous calculation of Γ_2 and Γ_3 provides

$$\Gamma_k = Mg_k \qquad (k = 1,2,3) \quad . \tag{5.13}$$

We apply Equation (5.13) to find the force of interaction between two point sources: M_1 at (0,0,0) and M_2 at (0,0,D). According to Equation (5.10), the field intensity created by M_1 at (0,0,D) is:

$$g_1 = \left(\frac{\partial\varphi}{\partial x_1}\right)_{x_1=D} = \frac{fM_1}{D^2}\ , \quad g_2 = g_3 = 0 \tag{5.14}$$

Hence, based on Equations (5.13), the driving force from source M_1 applied to M_2 is:

$$F_1 = -\frac{f\ M_1 M_2}{D^2} \qquad (F_2 = F_3 = 0) \quad . \tag{5.15}$$

Let us choose M in Equations (5.10) - (5.15) to be equal to the inertial mass, M_{iN},

$$M = M_{iN} \quad . \tag{5.16}$$

The inertial mass is defined by Newton's law as

$$M_{iN} = \frac{F_1}{a} \quad , \tag{5.17}$$

where a is the acceleration produced by force F_1. Measuring a and F_1 (e.g., by elastic spring deformation) yields M_{iN}. From here, by the comparison of Equations (5.13) and (5.17), we deduce that g_k is the acceleration of the free motion of unit mass in the gravitation field, which can be measured. One measurement of the gravitational force between two point masses enables one to calculate f using Equation (5.15), which becomes the well-known Newton law of gravitation (f is the universal gravitational constant).

In the classic non-relativistic approach to gravitation, the identity of the inertial and gravitational masses is a separate postulate. In the present approach, this is the issue of our option of the gravitational constant f. From the viewpoint of energy conservation law, the term $\varphi_{,i}\varphi_{,i}\ /\ (4\pi f\ )$ represents the potential energy of the gravitational field per unit volume, and the term $\varphi_{,i}\varphi_{,k} n_i\ /\ (4\pi f\ )$ the work rate of gravitational stresses. From the viewpoint of the momentum conservation law, these terms can also be interpreted as the specific momentum and gravitation stress tensor, respectively.

An analogous approach is possible for the construction of the relativistic field theory of gravitation (Einstein's theory of gravitation is geometrical, therefore within its framework it is difficult to introduce invariant integrals).

Electromagnetic field

Consider the stationary electromagnetic field in a medium with zero conductivity, ignoring the medium deformation. The field equations can be specified in the form of the following invariant integrals expressing the energy and momentum conservation laws:

$$\int_{\Sigma} \left(Wn_k - D_i n_i E_k - H_k B_i n_i \right) d\Sigma = 0 \quad , \qquad (i, k = 1,2,3) \tag{5.18a}$$

$$\lim_{\Sigma \to 0} \int_{\Sigma} \left(Wn_k - D_i n_i E_k - H_k B_i n_i \right) d\Sigma = \Gamma_k \quad , \tag{5.18b}$$

and the constitutive equations,

$$H_i = \frac{\partial U}{\partial B_i}, \quad E_i = \frac{\partial U}{\partial D_i}, \quad U = U(D_i, B_i) \quad . \tag{5.18c}$$

Here **D, E, H** and **B** are the field vectors, U is the potential energy of the field per unit volume (only reversible thermodynamic processes are here considered), and W is the function defined by

$$W(E_i, H_i) = U(D_i, B_i) + E_i D_i + H_i B_i \quad , \tag{5.19}$$

so that

$$D_i = \frac{\partial W}{\partial E_i} \quad , \quad B_i = \frac{\partial W}{\partial H_i} \quad . \tag{5.20}$$

The quantity Γ_k in (5.18b) is the energy of the field spent to move the singularity 0 unit length along the x_k axis, that is, the kth component of the driving force. Let us show that Maxwell's equations follow from the invariant integrals (5.18a). To the end, convert (5.18a) using the divergence theorem:

$$\begin{aligned} &\int_{\Sigma} \left(Wn_k - D_i n_i E_k - H_k B_i n_i \right) d\Sigma = \int_V \left[W,_k - \left(E_k D_i \right),_i - \left(H_k B_i \right),_i \right] dV = \\ &= \int_V \left[\frac{\partial W}{\partial E_i} E_{i,k} + \frac{\partial W}{\partial H_i} H_{i,k} - E_k \, D_{i,i} - E_{k,i} \, D_i - H_k B_{i,i} - H_{k,i} B_i \right] dV = \\ &= \int_V \left[D_i \left(E_{i,k} - E_{k,i} \right) + B_i \left(H_{i,k} - H_{k,i} \right) - E_k \, D_{i,i} - H_k B_{i,i} \right] dV = 0 \quad . \end{aligned} \tag{5.21}$$

From here, because V, D_i, B_i, E_k and H_k are arbitrary, it follows that

$$E_{i,k} = E_{k,i} \; ; \quad H_{i,k} = H_{k,i} \; ; \quad D_{i,i} = 0 \; ; \quad B_{i,i} = 0 \quad ; \tag{5.22}$$

and thus

$$\text{curl}\,\mathbf{E} = 0,\ \text{curl}\,\mathbf{H} = 0,\ \text{div}\,\mathbf{D} = 0,\ \text{div}\,\mathbf{B} = 0\ . \tag{5.23}$$

These are Maxwell's equations in the stationary case.

For an electrostatic field in an isotropic linear dielectric, for which

$$\mathbf{H} = 0,\ \ \mathbf{D} = \varepsilon\mathbf{E} \tag{5.24}$$

where ε is the dielectric constant, Equations (5.18) reduce to

$$\varepsilon \int_{\Sigma} \left(\frac{1}{2} E_i E_i n_k - E_i n_i E_k \right) d\Sigma = 0 \quad , \tag{5.25a}$$

$$\varepsilon \lim_{\Sigma \to 0} \int_{S} \left(\frac{1}{2} E_i E_i n_k - E_i n_i E_k \right) d\Sigma = \Gamma_k \quad , \tag{5.25b}$$

where

$$E_i = -\varphi,_i \quad , \quad \varphi,_{ii} = 0 \quad , \qquad i,k = 1,2,3 \quad . \tag{5.26}$$

(φ is the electrostatic potential).

Consider a singularity at 0. We write the singular solution of Laplace's equation in the form

$$\varphi = \frac{q}{4\pi\varepsilon r} + E_i x_i \quad , \tag{5.27}$$

where q is a constant, and E_i is the intensity of the external electrostatic field at 0 based on (5.26). As with Equations (5.11) - (5.13), one can derive the force driving this singularity to be

$$\Gamma_k = q\ E_k \qquad\qquad (k = 1,2,3) \quad . \tag{5.27}$$

The quantity q (charge) can be measured using this equation. From here and Equation (5.27), it follows that the interaction force between charge q_1 at (0,0,0) and q_2 at (L,0,0) is equal to $q_1\ q_2\ L^{-2}$, which is Coulomb's law.

Invariant integrals for a non-stationary electromagnetic field for irreversible processes in deformable media were given in references[1,2]. Invariant integrals for relativistic physical fields in the Minkovsky space-time were derived in reference[3] for ideal and viscous relativistic fluids and relativistic heat flow in fluids.

Nonequilibrium thermodynamics

Consider a nonequilibrium stationary thermodynamic field in which the entropy production rate, $\dot{S}$, can be written in the following form of a dot product of vectors

$$\dot{S} = \mathbf{X}^{\alpha}\,\dot{\mathbf{Q}}^{\alpha} \quad , \qquad (\alpha = 1,2,\ldots,N) \quad , \tag{5.28}$$

where the repeated superscript α denotes summation. Here $\mathbf{X}^{\alpha}$ and $\dot{\mathbf{Q}}^{\alpha}$ are the generalized thermodynamic force and flow, respectively; they are vectors in the physical space $0(x_1x_2x_3)$. The number N is equal to the number of different physical state functions describing the thermodynamical process under consideration. The typical state functions are temperature, concentrations of components taking part in diffusion and/or chemical reactions, pressure of the liquid or gaseous phase in a porous medium and so on. Equations (5.28) hold for the local equilibrium condition. Classical stationary fields of temperature or concentration in heat conductivity or diffusion can serve as examples where Equations (5.28) can be used.

In these fields, the entropy of any volume is stationary. This conservation law can be written in the form of the invariant integral

$$\int_{\Sigma}\left(Sn_k - X_i^{\alpha} n_i Q_k^{\alpha}\right) d\Sigma = 0 \quad , \qquad (i,k = 1,2,3) \tag{5.29a}$$

$$\lim_{\Sigma\to 0}\int_{\Sigma}\left(Sn_k - X_i^{\alpha} n_i Q_k^{\alpha}\right) d\Sigma = \Gamma_k \quad . \quad (\alpha = 1,2\ldots,N) \tag{5.29b}$$

Here Γ_k is the entropy flow from the external field spent (produced or absorbed) to move the field singularity 0 unit length along the x_k axis. The lower indices in X_i^{α} and Q_i^{α} denote the components of $\mathbf{X}^{\alpha}$ and $\mathbf{Q}^{\alpha}$ along the x_i axis.

Convert Equation (5.29a) using the divergence theorem:

$$\int_{\Sigma}\left(Sn_k - X_i^{\alpha} n_i Q_k^{\alpha}\right) d\Sigma = \int_{V}\left[S_{,k} - \left(X_i^{\alpha} Q_k^{\alpha}\right)_{,i}\right] dV = \tag{5.30}$$

$$= \int_{V}\left[\frac{\partial S}{\partial Q_i^{\alpha}} Q_{i,k}^{\alpha} - X_{i,i}^{\alpha}\, Q_k^{\alpha} - X_i^{\alpha}\, Q_{k,i}^{\alpha}\right] dV = \int_{V}\left[X_i^{\alpha}\left(Q_{i,k}^{\alpha} - Q_{k,i}^{\alpha}\right) - X_{i,i}^{\alpha}\, Q_k^{\alpha}\right] dV = 0$$

From here, because V, X_i^{α} and Q_k^{α} are arbitrary, it follows that

$$X_{i,i}^{\alpha} = 0 \quad , \quad Q_{i,k}^{\alpha} = Q_{k,i}^{\alpha} \quad , \qquad (i,k = 1,2,3) \tag{5.31}$$

and hence,

$$\text{curl } \mathbf{Q}^{\alpha} = 0, \;\; Q_i^{\alpha} = \text{grad } \varphi^{\alpha}, \qquad (\alpha = 1,2,\ldots,N) \tag{5.32}$$

where φ^{α} are some potentials.

For the linear law,

$$\mathbf{X}^{\alpha} = A_{\alpha\beta}\, \mathbf{Q}^{\beta} \qquad (\alpha,\beta = 1,2,\ldots,N) \tag{5.33}$$

the coefficients $A_{\alpha\beta}$ should satisfy Onsager's principle of symmetry

$$A_{\alpha\beta} = A_{\beta\alpha} \quad . \tag{5.34}$$

Expressing $\mathbf{X}^{\alpha}$ in terms of $\mathbf{Q}^{\beta}$ by using (5.33), and then (5.32), we may write (5.31) as

$$\left(A_{\alpha\beta}\, \varphi,_i^{\beta}\right),_i = 0 \quad , \tag{5.35}$$

where i = 1,2,3 and α, β = 1,2,...,N. This is the system of N partial differential equations governing the local-equilibrium thermodynamic process of the N coupled phenomena.

The motion of point sources or sinks of entropy is controlled by the driving force which can be derived from Equations (5.29b) and (5.35), just like the driving force on the gravitational mass or charge.

Gas dynamics

Consider irrotational stationary isoentropic flow of an inviscid compressible gas. In this case, the governing equations of gas dynamics can be written in the form of the following invariant integrals[4] :

$$\int_{\Sigma} \rho \mathrm{v}_i n_i d\Sigma = 0 \quad , \qquad (i,j,k = 1,2,3) \quad . \tag{5.36a}$$

$$\int_{\Sigma} \left(\rho \mathrm{v}_k \mathrm{v}_j n_j + p n_k\right) d\Sigma = 0 \quad , \tag{5.36b}$$

$$\lim_{\Sigma \to 0} \int_{\Sigma} \left(\rho \mathrm{v}_k \mathrm{v}_j n_j + p n_k\right) d\Sigma = \Gamma_k \quad , \tag{5.36c}$$

$$\frac{p}{p_{\infty}} = \left(\frac{\rho}{\rho_{\infty}}\right)^{\kappa} \quad . \tag{5.36d}$$

Here ρ, p and v_i are the density, pressure and velocity components of the gas, p_∞ and ρ_∞ are the pressure and density in the unperturbed flow, and $\kappa = c_p/c_v$ is the ratio of the heat capacities of the gas.

Equation (5.36a) expresses the mass conservation law, and Equations (5.36b) and (5.36c) express both momentum (per unit time) and energy (per unit path length) conservation laws. The quantity Γ_k represents the dissipation of energy of the external flow at 0 per unit path length of 0, that is, the force applied to 0 by the external flow. Equation (5.36d) expresses the process assumption that there is no entropy exchange between the material gas particles.

From Equations (5.36), it is easy to derive the classic equation system of gas dynamics:

$$(\rho v_i)_{,i} = 0\ , \quad (\rho v_k v_j)_{,j} = -p_{,k}\ , \quad p/p_\infty = (\rho/\rho_\infty)^\kappa\ . \tag{5.37}$$

Moreover, the condition equations on the shock waves in gas,

$$[\rho v_i n_i] = 0\ , \quad [\rho v_i n_i v_k + p n_k] = 0\ , \tag{5.38}$$

follow from the invariant integrals (5.36a) and (5.36b), as well. Here, n_i are the components of the unit normal vector to the shock wave surface, and the brackets denote the discontinuity of the quantity in the brackets on this surface. (Remember that the reference frame is fixed to the shock wave).

From (5.36c), for plane problems when 0 is a point trace of a singular line, one can derive (Problem 5.6)

$$\Gamma_i = 2\pi\rho_\infty v_\infty^2 C_i \qquad (i = 1,2)\ , \tag{5.39}$$

where C_1 is the value of mass production ($C_1 > 0$) or absorption ($C_1 < 0$) at 0 per unit length of path and per unit length of singular line, and C_2 is the vortex intensity at 0 (circulation). The Joukovsky - Chaplygin theory of wing lift can easily be derived from (5.36) and (5.39) (Problem 5.6).

For incompressible fluid, instead of Equation (5.36d), we have ρ = const, and the invariant integrals have the form:

$$\int_\Sigma v_i n_i d\Sigma = 0\ , \quad \int_\Sigma \left(v_k v_i n_i - \frac{1}{2} v_i v_i n_k \right) d\Sigma = 0\ , \tag{5.40a}$$

$$\rho \lim_{\Sigma\to 0} \int_\Sigma \left(v_i n_i v_k - \frac{1}{2} v_i v_i n_k \right) d\Sigma = \Gamma_k \qquad (i,k = 1,2,3)\ . \tag{5.40b}$$

From here, one can find the classic equations of fluid dynamics

$$\mathrm{v}_i = \varphi,_i \; ; \qquad \varphi,_{ii} = 0 \; ; \qquad p = -\frac{1}{2}\rho \mathrm{v}_i \mathrm{v}_i + const. \quad , \tag{5.41}$$

where φ is the fluid potential.

Theory of elasticity

Consider the static process of small deformations of an elastic solid. The governing equations of the process (theory of elasticity) can be written in the form of the following invariant integrals:

$$\int_{\Sigma} \sigma_{ij} n_j \; d\Sigma = 0 \qquad (i, j, k = 1,2,3) \quad , \tag{5.42a}$$

$$\int_{\Sigma} \left(U n_k - \sigma_{ij} u_{i,k} n_j \right) d\Sigma = 0 \quad , \tag{5.42b}$$

$$\lim_{\Sigma \to O} \int_{\Sigma} \left(U n_k - \sigma_{ij} u_{i,k} n_j \right) d\Sigma = \Gamma_k \quad , \tag{5.42c}$$

$$\lim_{\Sigma \to O} \int_{\Sigma} \sigma_{ij} \; n_i \; d\Sigma = F_j \quad . \tag{5.42d}$$

Here u_i are the displacement vector components, σ_{ij} are the stress tensor components, U is the elastic potential of the solid per unit volume, F_i are the components of the concentrated force at 0, and Γ_i are the components of the energy flow rate from the external field into 0 (in other words, the change of the elastic energy of the system due to the motion of 0 along the x_i axis, respectively, per unit length). The quantities Γ_i are the components of the singularity-driving force of the point 0, if $F_1 = F_2 = F_3 = 0$.

Using the divergence theorem, one can derive from the invariant integrals (5.42) the partial differential equations of the theory of elasticity:

$$\sigma_{ij,j} = 0 \qquad (i, j = 1,2,3) \quad , \tag{5.43a}$$

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \quad , \tag{5.43b}$$

where
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} \right) \quad , \qquad U = U(\varepsilon_{ij}) \quad . \tag{5.43c}$$

When U is a quadratic function of strains ε_{ij}, Equations (5.43) represent the traditional linear theory of elasticity.

5.2. Point Inclusions

The interstitial atoms of carbon, nitrogen, nickel and other alloying elements left in the lattice of a parent metal after solidification of a liquid mixture, hydrogen atoms penetrating into the solid metal lattice, impurity atoms, etc. will be referred to as point inclusions, if the diameter of foreign atoms is greater than the interatomic spacing of the lattice. Point inclusions are able to move with respect to the lattice due to self-diffusion and external loading.

Point inclusions are modeled as centers of compression described by the following equations[5] :

$$u_R^S = \frac{1+\nu}{2E}\frac{qa^3}{R^2} \qquad (q > 0), \tag{5.44a}$$

$$\sigma_R^S = -\frac{qa^3}{R^3}, \qquad \sigma_\psi^S = \sigma_\theta^S = \frac{qa^3}{2R^3}. \tag{5.44b}$$

Here R, ψ, and θ are spherical polar coordinates, u_R^S ,σ_R^S , σ_ψ^S and σ_θ^S are the displacement and stresses, respectively, a is the radius of the kernel of the compression center, q is the pressure of the kernel upon the elastic solid, and E and ν are Young's modulus and Poisson's ratio of the solid, respectively.

For $q < 0$ Equations (5.44) provide the singularity referred to as a "vacancy" in the physical literature. Both point inclusions and "vacancies" are traditionally called point defects. All the mathematical content of the present Section holds also for the "vacancies".

Physicists use the "vacancy" model for characterizing a small void in a lattice, for example, a vacant atomic volume not occupied by a parent atom. However, this model of small voids is physically incorrect because a void cannot create a tensile stress σ_R on the boundary (of the spherical void). The correct model of small voids referred to as point holes is given in Section 5.3 below. Probably, for $q < 0$ the "vacancy" model and the present theory can be applied to a foregn atom whose diameter is a little less than the void diameter so that the atom can create the cohesion force necessary to maintain the tensile stress σ_R on the void boundary.

It is easy to show, by means of Equations (5.42c) and (5.44), that the inclusion-driving force is zero for a uniform external stress field. Let us assume now that the unperturbed elastic field is non-uniform and is of the following form:

$$\begin{gathered} \sigma_{33}^0 = Ax_1, \quad u_1^0 = -\frac{\nu A}{2E}\left(x_1^2 - x_2^2 + \frac{1}{\nu}x_3^2\right), \\ u_2^0 = -\frac{\nu A}{E}x_1x_2, \quad u_3^0 = \frac{A}{E}x_1x_3 \end{gathered} \tag{5.45}$$

(A is a given constant).

Let us calculate the driving force which is, evidently, directed along the x_1 axis in this case. Using the invariance of the integral in Equation (5.42b), we take Σ as the parallelepiped formed by the faces: $x_3 = \pm\,\delta$, $x_1 = \pm\,L$, $x_2 = \pm\,L$ when $\delta/L \to 0$, $\delta \to \infty$, and $L \to \infty$. Using the rule of Γ-integration[6] yields

$$\Gamma_1 = -2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\sigma_{33}^{S}u_{3,1}^{0} + \sigma_{33}^{0}u_{3,1}^{S} + \sigma_{12}^{S}u_{2,1}^{0} + \sigma_{13}^{S}u_{3,1}^{0}\right)dx_1dx_2 \qquad \left(x_3 = \delta\right) \tag{5.46}$$

Here, we also used the symmetry with respect to the plane $x_3 = 0$ and the following relationships: $n_3 = 1$ for $x_3 = +\delta$, $n_3 = -1$ for $x_3 = -\delta$, $\sigma_{13}^{0} = \sigma_{23}^{0} = 0$ for $x_3 = \pm\delta$.
According to Equation (5.44), we have:

$$\sigma_{33}^{s} = \sigma_{z}^{s} = -\frac{qa^3}{4R^3}\left(1+3\cos 2\psi\right), \qquad \tau_{rz}^{s} = -\frac{3qa^3}{4R^3}\sin 2\psi \quad ,$$
$$\sigma_{13} = \frac{x_1}{r}\,\tau_{rz}, \qquad \sigma_{23} = \frac{x_2}{r}\,\tau_{rz}, \qquad u_{3,1}^{s} = -3\frac{1+\nu}{2E}\,qa^3\,\frac{x_1 z}{R^5} \quad . \tag{5.47}$$
$$\left(z = x_3\ , r^2 = x_1^2 + x_2^2\ , R^2 = r^2 + z^2\right)$$

Here, r, θ and z are cylindrical coordinates (ψ is measured from the z axis).

Using the equations (for $x_3 = \delta$),

$$R^2 = r^2 + \delta^2, \quad \delta/R = \cos\psi, \quad r/R = \sin\psi, \quad x_1/r = \cos\theta,$$

$$r\,dr = R\,dR = R^2\tan\psi\,d\psi, \quad dx_1\ dx_2 = r\,dr\,d\theta,$$

calculate integral (5.46) by means of (5.45) and (5.47):

$$\Gamma_1 = -\frac{2A}{E}\int_0^{\infty}\int_0^{2\pi}\left(\delta\sigma_z^s - \nu r\tau_{rz}^s + Ex_1u_{3,1}^s\right)r\,dr\,d\theta =$$
$$= \pi\frac{A}{E}qa^3\int_0^{\pi/2}\left[\sin\psi\left(1+3\cos 2\psi\right) - 3\nu\sin\psi\ \tan\psi\ \sin 2\psi + \right. \tag{5.48}$$
$$\left. +3\left(1+\nu\right)\ \sin^3\psi\right]d\psi = 2\pi\,\frac{1-\nu}{E}\,Aqa^3$$

To verify and clarify this result, we performed the exact calculation of Γ_1, by means of (5.42), over the integration sphere Σ, $R = R_o$, in which the elastic field is the sum of the fields described by Equations (5.44) and (5.45). After some cumbersome calculations, we found the following equation:

$$\Gamma_1 = 2\pi \frac{1-\nu}{E} Aqa^3 \left\{ 1 + \frac{pR_o^3}{qa^3(1-\nu)} \left[1 - \frac{5}{8}\nu \left(1 + \frac{3\pi}{128} \right) \right] \right\} \tag{5.49}$$

As seen, if

$$qa^3 >> pR_o^3 \quad , \tag{5.50}$$

the equation (5.49) reduces to (5.48). The condition equation (5.50) explains the physical meaning of the method of the asymptotic Γ-integration[6].

In another particular case of the non-uniform external field,

$$\sigma_{11}^o = Bx_1 \ , \sigma_{13}^o = \sigma_{31}^o = -Bx_3 , \tag{5.51}$$
$$u_1^o = \frac{B}{2E}\left[x_1^2 + \nu x_2^2 - (2+\nu)x_3^2\right], \quad u_2^o = -\frac{\nu B}{E}x_1 x_2 , \quad u_3^o = -\frac{\nu B}{E}x_1 x_3 ,$$

a similar calculation yields

$$\Gamma_1 = 2\pi \frac{1-\nu}{E} Bqa^3 , \qquad \Gamma_2 = \Gamma_3 = 0 \quad . \tag{5.52}$$

Let us also write the results of calculations for two more cases:

$$(i) \quad \sigma_{23}^0 = Cx_2 , \quad \sigma_{13}^0 = -Cx_1 , \quad u_1^0 = u_2^0 = 0 , \quad u_3^0 = \frac{C}{2G}\left(x_2^2 - x_1^2\right); \tag{5.53}$$

$$(ii) \quad \sigma_{23}^0 = Dx_1 , \quad u_1^0 = -\frac{D}{2G}x_2 x_3 , \quad u_2^0 = \frac{D}{2G}x_1 x_3 , \quad u_3^0 = \frac{D}{2G}x_1 x_2 \quad . \tag{5.54}$$

(G is shear modulus, C and D are constants). In both cases, we obtain $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. According to the basic rule of Γ-integration[6], the driving force is also equal to zero when the external stresses are arbitrary polynomials of x_1, x_2 and x_3 of degree 2 or more.

We can obtain the general case of an arbitrary non-uniform field of external stresses σ_{ij}^0 (x_1, x_2, x_3) by linear superposition of the particular cases considered above, after expanding the functions σ_{ij}^0 in Taylor series (in spite of the fact that the integrand in Equation (5.42) is non-linear). The final result is as follows[7,8] :

$$\Gamma_k = \lambda \frac{\Delta}{E} \frac{\partial \sigma}{\partial x_k} \qquad (k = 1,2,3) \quad . \tag{5.55}$$

Here

$$\Delta = qa^3, \quad \sigma = \sigma_{11}{}^0 + \sigma_{22}{}^0 + \sigma_{33}{}^0, \quad \lambda = 2\pi(1-\nu).$$

Equation (5.55) may also be derived by the help of equations (5.5) and (8.9) in the work[9] by Eshelby. Thus, the inclusion-driving force is directly proportional to the gradient of the first invariant of the external stress tensor σ_{ij}^{0} (x_1, x_2, x_3). The elegant formula (5.55) is analogous to that of Peach-Koehler in the theory of dislocations and that of Irwin in the theory of cracks. Let us discuss some basic problems of point inclusions by means of Equation (5.55). In the studies of the trajectories of inclusions, we assume that there are no preferential canals for the inclusions in the solid.

Interaction between inclusions

Let one of the inclusions be at the origin of coordinates. According to Equation (5.44), we have σ = 0 in the entire space in this case. Therefore, it follows from Equation (5.55) that the inclusions do not interact in unbounded space.

Interaction of inclusions with a dislocation

Let the front of an edge dislocation coincide with the x_3 axis, the Burgers vector being directed along the x_1 axis. The σ for the dislocation field is equal to

$$\sigma = \frac{BEx_2}{2\pi(1-\nu)\left(x_1^2 + x_2^2\right)} \tag{5.56}$$

(B is the magnitude of the Burgers vector). Let us place an inclusion at point (x_1, x_2, 0) and calculate the driving force acting upon it by means of Equations (5.55) and (5.56):

$$\Gamma_1 = -\frac{2\Delta Bx_1x_2}{\left(x_1^2 + x_2^2\right)^2}, \qquad \Gamma_2 = \frac{\Delta B\left(x_1^2 - x_2^2\right)}{(1-\upsilon)\left(x_1^2 + x_2^2\right)^2} \quad . \tag{5.57}$$

The path of the moveable inclusion is a solution of the following equation:

$$\frac{dx_2}{dx_1} = \frac{\Gamma_2}{\Gamma_1} = \frac{x_2^2 - x_1^2}{2x_1\ x_2}. \tag{5.58}$$

The general solution of Equation (5.58) is :

$$x_1^2 + x_2^2 = Cx_1 \quad (C = const.) \quad . \tag{5.59}$$

The family of curves in Equation (5.59) is the set of circles centered on the x_1 axis and tangent to the x_2 axis at the origin. According to Equation (5.57), the inclusions move in a clockwise direction for $x_1 < 0$ and $B > 0$, and in a counterclockwise direction for $x_1 > 0$,

$B > 0$. Hence, the inclusions are attracted to the core of the edge dislocation in the stretched region.

If a continuous cloud of inclusions is distributed in the space with the same dislocation, then in accordance with Equation (5.57), the dislocation-driving force is

$$\Gamma_1 = B\tau_\infty + 2\Delta B \iint \frac{x_1 x_2 N(x_1, x_2) dx_1 dx_2}{\left(x_1^2 + x_2^2\right)^2} \qquad \left(\delta = \frac{4}{3}\pi a^3 \, N\rho_{inc}\right). \tag{5.60}$$

Here τ_∞ is the external shear stress on the plane $x_2 = 0$, δ and N are the mass and number of inclusions per unit volume, respectively, and ρ_{inc} is the density of inclusion material. The first and second terms in Equation (5.60) are, respectively, the Peach-Koehler force and the force induced by the cloud of inclusions; the latter describes the arresting effect of inclusions on the dislocation.

Interaction of inclusions with a crack

Suppose an inclusion is at a certain point (x_1, x_2, 0) in the vicinity of the opening mode crack along $x_2 = 0$, $x_1 < 0$, $-\infty < x_3 < \infty$. For this case, we find

$$\sigma = \frac{\sqrt{2}(1+\nu)K_I}{\sqrt{\pi r}} \cos\frac{\theta}{2} \qquad (r^2 = x_1^2 + x_2^2, \quad tan\theta = x_2 / x_1) \tag{5.61}$$

where K_I is the stress intensity factor, r and θ are the polar coordinates of the inclusion in the plane $x_3 = 0$.

We find the components of the inclusion-driving force by means of Equations (5.55) and (5.61):

$$\Gamma_r = -\frac{\lambda\Delta(1+\nu)\,K_I}{E\sqrt{2\pi}\,r^{3/2}} \cos\frac{\theta}{2} \quad ,$$
$$\Gamma_\theta = -\frac{\lambda\Delta(1+\nu)\,K_I}{E\sqrt{2\pi}\,r^{3/2}} \sin\frac{\theta}{2} \quad , \qquad \Gamma_3 = 0. \tag{5.62}$$

From this, the solution of the following equation provides the trajectories of the inclusions:

$$r\frac{d\theta}{dr} = \frac{\Gamma_\theta}{\Gamma_r} = tan\frac{\theta}{2} \quad . \tag{5.63}$$

Hence, the family of the trajectories will be as follows:

$$\sqrt{r} = C \sin\frac{\theta}{2} \qquad (C = const.) \quad . \tag{5.64}$$

These are closed oval curves tangential to the x_1 axis at the origin and symmetrical with respect to the x_1 axis. The inclusions move along the ovals clockwise for $x_2 > 0$ and counterclockwise for $x_2 < 0$. Thus, the inclusions are attracted to the crack tip, namely to the stretched pre-fractured region of the crack's continuation. The inclusions near the crack front cause an increase of the crack-driving force, since

$$\Gamma_1 = \Gamma_r \cos\theta - \Gamma_\theta \sin\theta \,. \tag{5.65}$$

This means that inclusions embrittle the material. Furthermore, inclusions arrest dislocations near the crack tip which increases the embrittlement effect.

For a continuous cloud of inclusions in a material with the same crack, the crack-driving force is equal to

$$\Gamma_1 = \frac{1-\nu^2}{E} K_1^2 + \frac{\lambda\Delta(1+\nu) K_I}{E\sqrt{2\pi}} \iint \frac{\cos\frac{3\theta}{2} N(r,\theta)}{r^{1/2}} dr\, d\theta \tag{5.66}$$

Here, the first and second terms are, respectively, the Irwin force originated by external loads and the force induced by the cloud of inclusions.

Interaction of inclusions with a spherical cavity

Suppose an infinite elastic space with a spherical cavity of radius R_a is subjected to uniaxial extension by the stress $\sigma_z = p$. The center of the sphere is assumed to coincide with the origin of the cylindrical coordinate frame. The surface of the sphere is free of external loads. In this case, the first invariant of the stress tensor[5] is equal to

$$\sigma = p + \frac{5p(1+\nu)}{7-5\nu} R_a^3 \frac{\partial}{\partial z}\left(\frac{z}{R^3}\right) = p + 5p\, R_a^3 \frac{1+\nu}{7-5\nu} \frac{r^2-2z^2}{R^5} \tag{5.67}$$

$$(R^2 = r^2 + z^2).$$

Consider an inclusion at a point. According to Equations (5.55) and (5.67), the components of the inclusion-driving force are

$$\Gamma_r = \frac{15\lambda p\Delta(1+\nu)}{E(7-5\nu)} \frac{rR_a^3(4z^2-r^2)}{R^7} \,,$$
$$\Gamma_z = \frac{15\lambda p\Delta(1+\nu)}{E(7-5\nu)} \frac{zR_a^3(2z^2-3r^2)}{R^7} \,. \tag{5.68}$$

(r and z are the coordinates of the inclusion).

The velocity of the moveable inclusion and the driving force have the same direction. Therefore, the path of the inclusion is an integral curve of the following equation:

$$\frac{dr}{dz} = \frac{r\left(4z^2 - r^2\right)}{z\left(2z^2 - 3r^2\right)} \quad . \tag{5.69}$$

The family of the integral curves of this equation is as follows:

$$\begin{aligned} &2CR^3 = \sin 2\psi \quad or \quad zr = CR^5 \\ &\left(R^2 = z^2 + r^2 , \quad z/r = \cos\psi , \quad C = const\right) \quad . \end{aligned} \tag{5.70}$$

The trajectories of inclusions are qualitatively depicted in Figure 5.1. As seen, the inclusions travel to the zone of the highest tensile stresses near the cavity, which implies a hardening effect.

The continuum theory of inclusions

Consider a solid with a large number of point inclusions, that is, a cloud of inclusions. Within the framework of the asymptotic approach involved in Equation (5.55), the interaction of inclusions can be ignored. We study the self-diffusion and drift transport of the inclusions.

As generally accepted in the linear theory of diffusion[10], the drift rate of inclusions, $\mathbf{V_d}$, is directly proportional to the driving force, $\mathbf{\Gamma}$,

$$\mathbf{V}_{\boldsymbol{d}} = \eta\,\mathbf{\Gamma} \quad , \tag{5.71}$$

where η is an empirical coefficient of mobility of inclusions.

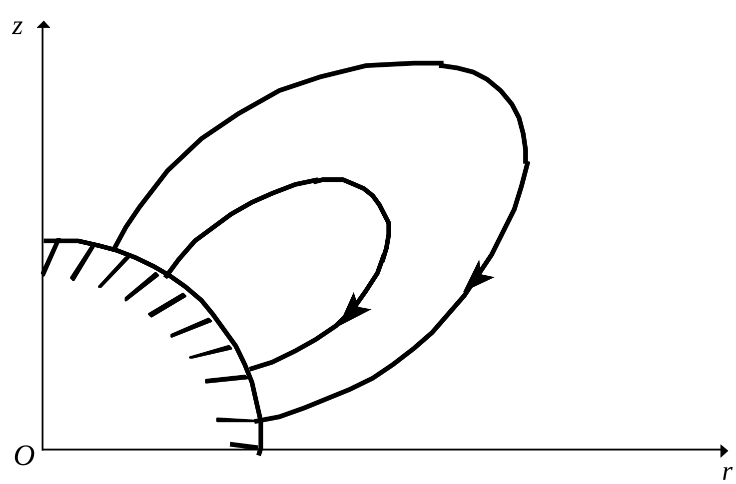

Figure 5.1. Schematic paths of moveable inclusions in the neighborhood of a fixed spherical hole

The inclusion transport equation will thus be

$$\frac{\partial \delta}{\partial t} = \left(D\delta_{,k}\right)_{,k} - \left(\eta \frac{\lambda \Delta}{E} \delta\sigma_{,k}\right)_{,k} \qquad (k = 1,2,3) \quad . \qquad (5.72)$$

Here t is time, D is the self-diffusion coefficient, δ is the mass of inclusions per unit volume. The quantities η and D depend on temperature. For inclusions of the same type coefficients D, η and Δ are identical. Therefore, the total number of equations equals the number of different types of inclusions. The influence of other external fields (such as electric, heat and chemical ones) upon the transport of inclusions can be treated by methods using the relevant linear or non-linear governing equations of irreversible thermodynamics similar to Equation (5.71). The effect of new fields would result in additional terms in Equation (5.72).

As an illustration, we find the solution to Equation (5.72) for the strip $0 < x_1 < d$ subjected to pure bending by the moment M (per unit length). From the elastic solution, one obtains

$$\sigma_{,2} = \sigma_{,3} = 0, \qquad \sigma_{,1} = 12(1+\nu)Md^{-3}. \qquad (5.73a)$$

The stationary equilibrium concentration of inclusions in the strip under the action of bending is found from the solution of Equation (5.72) with $\partial\delta/\partial t = 0$,

$$\delta = C_o \left(e^{\mu d} - e^{\mu x_1}\right), \quad \text{where} \quad \mu = \frac{12\,(1+\nu)M\eta\lambda\Delta}{EDd^3} \quad . \qquad (5.73b)$$

The constant C_0 is determined by the total mass m_0 of moveable inclusions:

$$C_0 = \frac{\mu m_0}{e^{\mu d} - 1}, \quad \text{where} \quad \int_0^d \delta\, dx_1 = m_0 \qquad (5.74)$$

This solution allows one to predict the distribution of material properties across the strip thickness, if it is known how the corresponding property depends on the concentration of inclusions.

According to the principle of superposition, the internal stresses induced by a continuous cloud of inclusions are as follows:

$$\sigma_{ik}\left(x_1, x_2, x_3\right) = \int \delta\left(x_1', x_2', x_3'\right)\sigma_{ik}'\left(x_1 - x_1',\, x_2 - x_2',\, x_3 - x_3'\right)dx_1'\, dx_2'\, dx_3' \quad . \qquad (5.75)$$

Here, the stresses σ_{ik}' are determined by the stresses (5.44) in the corresponding coordinate system. The δ in this case is defined by (5.73) and (5.74).

5.3. Point Holes

Voids in a crystal lattice, small bubbles and cavities always present in materials are referred to as point holes. If they are sufficiently small or when there are some channels in the structure of the material, the holes can move relative to the lattice under the action of driving forces. Physically, the smallest point hole is a vacant atomic volume in a lattice not occupied by an atom.

Let us model a hole by a spherical cavity of radius r_0 with the surface of the cavity being free of external loads. The distance, R, between holes is assumed to be large compared to r_0. Actually, it is accurate to assume that $R > 4r_0$, that is, the porosity of the material involved should be less than 0.07. This assumption allows us to consider holes as point sources of asymptotically singular perturbation, or, in other words, as some quasi-particles[7,8]. Like any point source of perturbation, a hole is acted upon by a driving force, whose components are determined by Equation (5.42c).

The displacements u_r and u_z brought about by extension of an infinite elastic space with a spherical cavity of radius r_0 are[5] :

$$
\begin{aligned}
u_r &= -\frac{\nu p}{E} r + \frac{prr_0^3}{4GR^3}\left[-\frac{z^2}{R^2} - \frac{4-10\nu}{7-5\nu}\left(1-\frac{3z^2}{R^2}\right) + \right.\\
&\left. + \quad \frac{3r_0^2 - (2+5\nu)R^2}{(7-5\nu)R^2}\left(1-\frac{5z^2}{R^2}\right)\right] \qquad \left(R^2 = r^2 + z^2\right) \quad ,
\end{aligned} \tag{5.76a}
$$

$$
\begin{aligned}
u_z &= \frac{p}{E} z + \frac{pzr_0^3}{4GR^3}\left[\frac{2}{3} - \frac{z^2}{R^2} - \frac{4-10\nu}{7-5\nu} \left(1-\frac{3z^2}{R^2}\right) + \right.\\
&\left. + \frac{(2+5\nu)R^2 - 3r_0^2}{(7-5\nu)R^2} \left(-3+\frac{5z^2}{R^2}\right)\right]. \qquad \left(G = \frac{E}{2(1+\nu)}\right)
\end{aligned} \tag{5.76b}
$$

Here, r and z are cylindrical coordinates whose origin is at the center of the sphere and, p is the tensile stress σ_z at infinity.The stresses are determined from (5.76) by Hooke's law.

Using the invariance of Γ_k with respect to Σ, we contract Σ onto the surface of the spherical cavity. Making use of Equations (5.42) and (5.76), we can prove that $\Gamma_k = 0$.

We now assume that the unperturbed external field has a non-uniform component $\sigma^0_{33} = Ax_1$, and that $Ar_0 << p$. The following displacements correspond to this component:

$$
\begin{aligned}
u_1^0 &= -\frac{\nu A}{2E}\left(x_1^2 - x_2^2 + \frac{1}{\nu}x_3^2\right), \quad u_2^0 = -\frac{\nu A}{E}x_1 x_2 ,\\
u_3^0 &= \frac{A}{E}x_1 x_3 \qquad \left(x_3 = z, \quad x_1^2 + x_2^2 = r^2\right).
\end{aligned} \tag{5.77}
$$

Using the invariance of Γ integrals in Equation (5.42), we take integration surface Σ, to be the parallelepiped $x_1 = \pm L$, $x_2 = \pm L$, $x_3 = \pm \delta$, with $\delta/L \to 0$, $\delta \to \infty$, $L \to \infty$. From this, taking account of the rule of Γ integration[6], we have [1-3]

$$\Gamma_1 = -2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\sigma_{33}^0 \quad u_{3,1}^s + \sigma_{33}^s \quad u_{3,1}^0 + \sigma_{23}^s \, u_{2,1}^0 + \sigma_{13}^s \, u_{1,1}^0\right) dx_1 \, dx_2 \,, \tag{5.78}$$

along $x_3 = \delta \to \infty$.

The singular field of perturbation (with the superscript s) is determined with the help of Equation (5.76). The calculation yields the following value for the hole-driving force:

$$\Gamma_1 = \frac{\lambda_2 pA}{E} r_0{}^3 \quad (\Gamma_2 = \Gamma_3 = 0), \tag{5.79a}$$

where

$$\lambda_2 = \frac{2\pi}{3} \; \frac{59 - 86\nu - 25\nu^2}{7 - 5\nu} \tag{5.79b}$$

In the general case of an arbitrary non-uniform field of external stresses σ^0_{ij} (x_1,x_2,x_3) satisfying the condition,

$$\left|\sigma_{ij}^0\right| >> r_0 \left|\sigma_{ij,k}^0\right| \;, \tag{5.80}$$

the stresses in the singular solution of the hole-type singularity are directly proportional to the stresses σ^0_{ij} of the hole-unperturbed field. According to the Γ integration rule[6], the hole-driving force is directly proportional to the stresses σ^0_{ij} and their gradients, $\sigma^0_{ij,k}$. Therefore, the interaction energy, U, of the hole with the external stress field is a positive quadratic function of stresses, so that, in the general case of anisotropic solids and holes of arbitrary shape, U and Γ_k have the form:

$$U = C_{ijmn}\sigma_{ij}^0\sigma_{mn}^0 \quad , \tag{5.81a}$$

$$\Gamma_k = -\frac{\partial U}{\partial x_k}, \tag{5.81b}$$

where the constants C_{ijmn} satisfy

$$C_{ijmn} = C_{jimn} = C_{ijnm} = C_{mnij} \qquad (i, j, k, m, n = 1,2,3). \tag{5.81c}$$

For a spherical hole and isotropic elastic solid, U depends only on the first and second invariants, σ and I, of the external stress tensor:

$$U = r_0^3 \, E^{-1}\left(\alpha\sigma^2 + \beta I\right), \qquad \Gamma_k = -\frac{\partial U}{\partial x_k}, \tag{5.82a}$$

where

$$\sigma = \sigma_{11}^0 + \sigma_{22}^0 + \sigma_{33}^0,$$
$$I = \sigma_{11}^0\sigma_{22}^0 + \sigma_{22}^0\,\sigma_{33}^0 + \sigma_{11}^0\,\sigma_{33}^0 - \left(\sigma_{12}^0\right)^2 - \left(\sigma_{13}^0\right)^2 - \left(\sigma_{23}^0\right)^2. \tag{5.82b}$$

Here, α and β are some material constants to be defined.

For omni-directional uniform extension,

$$\sigma_{ij} = \frac{1}{3}\sigma\delta_{ij} \quad \text{where} \quad \sigma > 0, \tag{5.83}$$

the hole is the compression center defined by Equation (5.44), whose interaction energy, according to Equation (5.55), is equal to

$$-\lambda\,\frac{\sigma^2 r_0^3}{6E}. \tag{5.84}$$

From Equations (5.82) and (5.84), it follows that

$$3\alpha + \beta = -\pi(1-\nu). \tag{5.85}$$

For another example, take $\sigma_{33}^0 = p$ and $\sigma_{33,1}^0 = A$, and the remaining stresses equal zero. The interaction energy given by Equation (5.79) is $-\lambda_2\, x_1 pAr_0^3 E^{-1}$ and $-2\alpha pA\, x_1\, r_0^3\, E^{-1}$ given by Equation (5.82). Comparison of both expressions yields $2\alpha = -\lambda_2$, and matching this result with (5.85), one can find α and β:

$$\alpha = -\frac{1}{2}\lambda_2, \quad \beta = \frac{3}{2}\lambda_2 - \pi(1-\nu), \qquad (\alpha < 0, \beta > 0), \tag{5.86}$$

where λ_2 is given by (5.79b) in terms of ν.

According to the basic equation (5.82a), the hole-driving force is directed to move the hole to a more stressed zone, independent of the sign of the stresses. Therefore, the hole behaves absolutely different, even qualitatively, from the classical vacancy described by Equations (5.44) and (5.55) for q <0. For example, when the external field is close to the omni-directional extension, the hole behaves rather like an inclusion.

The continuum theory of holes

Suppose an elastic medium contains a very large number of small holes which may be described by a continuous distribution of a cloud of holes. The movement of holes is assumed to be occurring under the action of external loads and heat fluctuations

The flux of holes is obviously equivalent to the flux of matter in the opposite direction. Therefore, the hole mass transport equation is:

$$\frac{\partial \varepsilon}{\partial t} = \left(D\varepsilon_{,i} \right)_{,i} + \left(\eta \varepsilon U_{,i} \right)_{,i} \quad , \tag{5.87a}$$

where

$$\Gamma_i = -U_{,i} \quad , \quad \varepsilon = 1 - \frac{\rho}{\rho_o} \quad , \qquad \mathbf{V}_d = \eta \, \mathbf{\Gamma} \quad . \tag{5.87b}$$

Here ε is the porosity, $\rho\,(x_1, x_2, x_3, t)$ is the sought-for density of the material, D is the self-diffusion coefficient, η is the mobility coefficient of holes, and U is defined by Equation (5.82a). The second term in Equation (5.87a) is written using the assumption that the drift rate of holes is directly proportional to the hole-driving force[7,8].

Equation (5.87) should be supplemented with the equations of the theory of elasticity

$$\sigma^o_{ij,j} = 0, \quad \frac{1}{2} E\left(u_{i,j} + u_{j,i} \right) = (1 + \nu)\sigma^o_{ij} - \nu \sigma^o_{kk} \delta_{ij} \ . \tag{5.88}$$

For porous materials, it is natural to assume the following rule of mixture:

$$E = E_{max} \frac{\rho}{\rho_{max}} \tag{5.89}$$

(E_{max} and ρ_{max} correspond to zero porosity).

The closed system of Equations (5.87) - (5.89) allows one to study evolution of the cloud of holes and the progressive formation of weakened crack-like zones of high porosity. It may be important for studying creep and fracture of a material under high temperature.

Interaction between two holes

Consider a fixed hole of radius r_o with its center at the coordinate origin in an infinite space uniformly stretched along the z axis by stress $\sigma_z = p$. Let another hole of radius r_1 be placed in the field of singular perturbation of the first hole at a point with spherical coordinates R, ψ and θ ($r_1 << R$, ψ is measured from the z axis). Because of the non-uniformity of the total field, the latter hole is acted upon by the driving force with components Γ_R and Γ_ψ ($\Gamma_\theta = 0$).

After some calculations, Equations (5.76) and (5.82a) lead to the following simple equations

$$\Gamma_R = \frac{45\lambda p^2 r_o{}^3 r_1^3}{2(5-7\nu)ER^4}\left(-1+2\nu-6\nu\cos^2\psi+5\cos^4\psi\right) \quad ,$$
$$\Gamma_\psi = \frac{15\lambda p^2 r_o{}^3 r_1^3}{(7-5\nu)ER^4}\sin 2\psi\left(-1-\nu+5\cos^2\psi\right) \quad . \tag{5.90}$$

Let a cloud of small moveable holes be in the field of a large fixed hole at the coordinate origin, the interaction of small holes being ignored. According to Equation (5.90), the moveable holes are attracted, under the action of uniaxial extension-compression, to the cone $\psi = \psi_*$ where

$$\psi_* = arc\cos\sqrt{(1+\nu)/5}, \quad \text{so that} \quad \psi_* = 60 \pm 4^o \quad . \tag{5.91}$$

and then travel along this cone into the fixed hole (Figure 5.2). The exact paths of holes are determined if one integrates the following equation:

$$\frac{V_R}{V_\psi} = \frac{d\ln R}{d\psi} = \frac{\Gamma_R}{\Gamma_\psi} \quad , \tag{5.92}$$

where V_R and V_ψ are components of the velocity of holes.

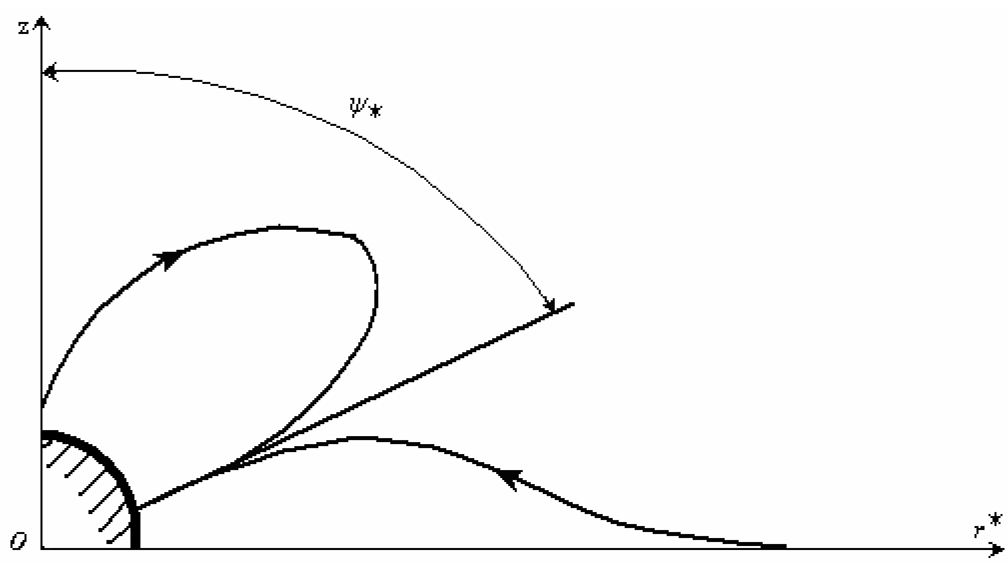

Figure 5.2. A schematic path of a moveable hole in the vicinity of a fixed spherical cavity

Because of the attraction of movable holes to the cone $\psi = \psi_*$, the planes $\psi = \psi_*$ far from the big hole become the place where the movable holes accumulate and form a loosen zone which develops to a shear fracture (under uniaxial load $\sigma_z = -\sigma$). The angle $\psi = \psi_*$ thus can be interpreted as the angle of internal friction of the material characterized by the friction coefficient

$$f = \frac{\tau_n}{\sigma_n} = cotan\,\psi_*$$

where $\tau_n = \sigma \sin\psi_* \cos\psi_*$, $\sigma_n = \sigma \sin^2 \psi_*$.

So, we have predicted the friction coefficient to be $f = 0.6 \pm 0.1$ which is the value of f which is most frequently observed in compression tests of various brittle rocks.

Interaction of holes with a crack front

Let a hole of radius r_0 be in the vicinity of an opening mode crack along $\theta = \pi$, $0 < r < \infty$, $-\infty < z < \infty$ where r, θ and z are cylindrical coordinates. The external stress field is as follows:

$$\begin{aligned}
\sigma_r^0 &= \frac{K_I}{\sqrt{2\pi r}}\left(cos\,\frac{\theta}{2} + \frac{1}{2}\, sin\,\theta\, sin\,\frac{\theta}{2} \right) , \\
\sigma_\theta^0 &= \frac{K_I}{\sqrt{2\pi r}}\left(cos\,\frac{\theta}{2} - \frac{1}{2}\, sin\,\theta\, sin\,\frac{\theta}{2} \right) , \qquad (5.93)\\
\tau_{r\theta}^0 &= \frac{K_I}{\sqrt{2\pi r}}\, sin\,\theta\, cos\,\frac{\theta}{2} , \quad \sigma_z = \frac{2\nu K_I}{\sqrt{2\pi r}}\, cos\,\frac{\theta}{2} .
\end{aligned}$$

(K_I is the stress intensity factor).

According to Equation (5.82a), the components of the driving force are:

$$\begin{aligned}
\Gamma_r &= -\frac{\partial U}{\partial r} , \quad \Gamma_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta} , \\
U &= \frac{r_0^3 K_I^2}{2\pi E r}\, cos^2\frac{\theta}{2}\left[4\alpha\,(1+\nu)^2 + \beta\left(4\nu + cos^2\,\frac{\theta}{2} \right)\right] , \qquad (5.94)\\
\sigma &= \frac{2\,(1+\nu)}{\sqrt{2\pi r}}\, K_I\, cos\,\frac{\theta}{2} , \\
I &= \frac{K_I^2}{2\pi r}\, cos^2\frac{\theta}{2}\left(4\nu + cos^2\frac{\theta}{2} \right) .
\end{aligned}$$

Taking Equations (5.93) and (5.94) into account, one can find

$$\Gamma_r = \frac{r_0^3 K_I^2}{2\pi E r^2} \cos^2\frac{\theta}{2}\left[4\alpha(1+\nu)^2 + \beta\left(4\nu + \cos^2\frac{\theta}{2}\right)\right] ,$$
$$\Gamma_\theta = \frac{r_0^3 K_I^2}{2\pi E r^2} \sin\theta\left[2\alpha(1+\nu)^2 + \beta\left(2\nu + \cos^2\frac{\theta}{2}\right)\right] . \tag{5.95}$$

Here r and θ are the hole coordinates.

The values of Γ_r and Γ_θ are negative for all θ in the range $0 < \theta < \pi$, that is, the hole is always attracted to the crack tip. The path of moveable holes is depicted qualitatively in Figure 5.3. The family of the paths is governed by the following equation:

$$\frac{V_r}{V_\theta} = \frac{d\ln r}{d\theta} = \frac{\Gamma_r}{\Gamma_\theta} . \tag{5.96}$$

The travel of holes into the crack tip leads to a reduction of fracture toughness and a subcritical creep-like growth of the crack. The holes concentrate on the continuation of the crack; their condensation at the crack tip allows one to predict the kinetics of slow crack growth due to the mechanism of migration/diffusion of holes in the material.

In addition, the holes change the crack-driving force. Let a cloud of holes be distributed in a material, so that there are N holes in unit volume, with porosity equal to $\varepsilon = 4\pi N r_0^3 / 3$. Adding up forces induced by external loads and cloud of holes, we get

$$\Gamma = \frac{1-\nu^2}{E} K_I^2 - \int_{\delta_0}^{L}\int_{-\pi}^{\pi} N\left(\Gamma_r \cos\theta - \Gamma_\theta \sin\theta\right) r\, dr\, d\theta . \tag{5.97}$$

Here, L is a characteristic length of a crack (or a body). The first term in Equation (5.97) represents the Irwin force originated by external loads, the second being the force induced by holes. For the case of uniform distribution of holes when N and ε are independent of r and θ, the second term in Equation (5.97) vanishes because of (5.95).

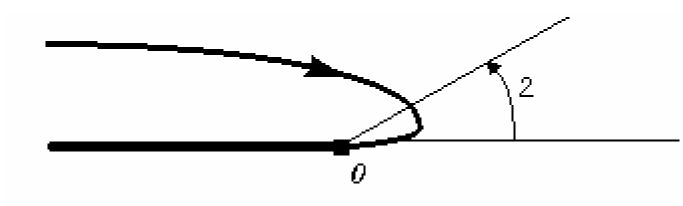

Figure 5.3. A schematic path of a moveable hole in the neighborhood of a fixed opening mode crack

Interaction of holes with a dislocation

Let a hole of radius r_0 be near the front of an edge dislocation, so that

$$\sigma_r^o = \sigma_\theta^o = \frac{BE \sin\theta}{4\pi(1-\nu^2)r}\ , \quad \tau_{r\theta}^o = -\frac{BE\cos\theta}{4\pi(1-\nu^2)\,r}\ , \quad \sigma_z^o = 2\nu\sigma_r^o \quad , (5.98)$$

where B is the value of the Burgers' vector. By means of Equations (5.94) and (5.98), we calculate the components of the hole-driving force

$$\Gamma_r = -\frac{r_o^3\ B^2 E}{8\pi^2\ (1-\nu^2)^2 r^3}\left[2(1+2\nu)(2\alpha+\beta)\sin^2\theta - \beta\right] \ ,$$
$$\Gamma_\theta = -\frac{r_o^3\ B^2 E}{8\pi^2\ (1-\nu^2)^2 r^3}\ \sin 2\theta\left[2\alpha(1+\nu)^2 + \beta(1+2\nu)\right] \ . \quad (5.99)$$

(r and θ are coordinates of the hole). From (5.99), it follows that the hole is always attracted to the core of the dislocation. The path of a moveable hole is qualitatively depicted in Figure 5.4.

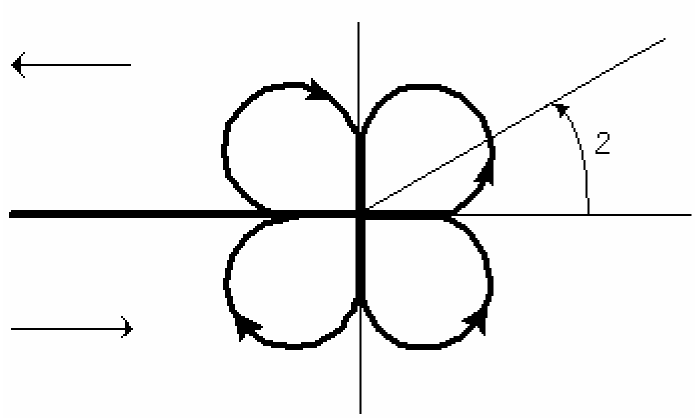

Figure 5.4. A schematic path of a moveable hole in the neighborhood of a fixed edge dislocation

If holes are continuously distributed in a material, the force Γ driving an edge dislocation is equal to

$$\Gamma = B\tau_\infty - \int_{r_d}^{L}\int_{-\pi}^{\pi} rN\left(\Gamma_r \cos\theta - \Gamma_\theta \sin\theta\right) dr\, d\theta \quad . \quad (5.100)$$

Here r_d is the radius of the dislocation core, L is a characteristic length of the body, and τ_∞ is an external shear stress on the plane of dislocation. The terms in Equation (5.100) represent the Peach-Koehler force and the force induced by a cloud of holes, respectively. For a uniform distribution of holes, the second term vanishes.

5.4. Conclusion

The invariant integral appears to be a very efficient tool for the study of driving forces and motion of miscellaneous point defects in solids.

The inclusion-type defects are important for the understanding of the hardening effect of a basic metal by alloying elements and their influence on fracture toughness. The inclusions were proved to move to stretched zones.

The hole-type defects are necessary for the understanding of creep in metals. The following features are proved to be characteristic of holes:

(*i*). Attraction and absorption of small holes by bigger ones;

(*ii*). Tendency to get collected along some planes that are inclined at an angle to the direction of uniaxial extension/compression;

(*iii*). Indifference with respect to the sign of external stress (extension or compression).

Driving forces constitute one side of the problem. The other side is the measurement of drag experienced by a point defect moving in a lattice. Knowledge of the threshold value of the driving force that is necessary for a point defect to overcome the resistance and move in the lattice is very important for the prediction of the response of a solid material to loading and temperature variations.

Some conceptual points of view treated in the present chapter are near to those developed in the recent books by Maugin[11-13].

The following problems constitute an essential part of the chapter and are designed for those who want to make creative use of the methods treated.

5.5. Problems

5.1. Find the invariant integrals for the stationary flow of viscous incompressible fluid. Derive the Navier-Stokes equations using the invariant integrals.

5.2. Find the drag of a slender semi-infinite penetrator in an elastic medium moving along its axis with zero friction. Consider both cracking and no cracking conditions.

5.3. Using the invariant integral, prove the following law of action and counteraction: if a field singularity, *A*, induces the force $\mathbf{\Gamma}$ acting upon another field singularity, *B*, then a force of the same magnitude, but opposite direction, $-\mathbf{\Gamma}$, acts from *B* upon *A*.

5.4. Prove that the cavity-driving force in an elastic medium is equal to zero for a cavity of arbitrary shape, if the stress field is uniform, that is, does not vary in space. This is an analogue of Euler's paradox in hydrodynamics.

5.5. Prove that the resultant force (drag) applied to a solid body moving steadily in an ideal compressible gas is equal to zero if there are no vortices, mass sinks and sources, cavities, and shock waves (Euler's paradox).

5.6. Consider the plane flow of an ideal compressible gas of density ρ_∞ over a profile moving at constant speed V_∞. Prove that the propelling force,

$$\Gamma_1 = 2\pi\rho_\infty V_\infty^2\, q, \tag{5.101}$$

appears when the profile surface absorbs the gas of mass q per unit length.

Prove that the lift force of a plane profile is equal to

$$\Gamma_2 = 2\pi\rho_\infty V_\infty^2 I_v \quad , \tag{5.102}$$

where I_v is the circulation (vortex intensity) at infinity produced by the profile (the Joukovsky-Chaplygin theorem).

5.7. Solve Problems 5.5 and 5.6 for an incompressible fluid.

5.8. Prove the transformation,

$$U_{,k}=(\sigma_{ij}\, u_{i,k})_{,j} \;=\; \sigma_{ij,\,j}\, u_{i,k} + \frac{1}{2}\sigma_{ij}\,(u_{i,jk} + u_{j,ik})\,, \tag{5.103}$$

and derive Equation (5.43) from the invariant integrals (5.42).

5.9. Consider a thin flattened rigid penetrator moving in an elastic medium at a constant speed. Prove that the drag is equal to zero, if: (*i*) there is no friction on the body surface; and (*ii*) there is no cracking or fracturing in the elastic medium.

5.10. As distinct from the conditions of Problem 5. 9, consider a crack formed by the flattened penetrator, but closed at a finite distance behind the penetrator. Prove that the drag is also equal to zero in this case.

5.11. Consider the steady-state motion of a semi-infinite cylindrical rigid body along its axis in gas or fluid. Show that Euler's paradox is not valid in this case and the drag is equal to

$$\Gamma_1 = \frac{1}{2}\rho V^2 \Sigma_c\,, \tag{5.104}$$

where V is the velocity, and Σ_c is the cross-section area of the cylinder.

5.12. Study interaction between a point hole and a point inclusion.

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